Synchronous Rendezvous of Very-Low-Range Wireless Agents

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***Abstract*— In this paper, we address the problem of temporal synchronization of a team of mobile agents on a set of rendezvous points defined by the nodes of a bipartite network. In particular, we assume very-low-range (VLR) wireless agents that travel along the edges of a bipartite network, and can only communicate with each other when they rendezvous at the nodes of the network. Temporal synchronization relies on distributed agreement protocols on the travel times and waiting times at the rendezvous points, and when integrated with the agents' motion, results in a hybrid multi-agent coordination scheme. We show that the resulting dynamic communication network is almost always eventually connected and, therefore, synchronous rendezvous of the agents on the nodes of the underlying path network is guaranteed. Unlike prior work on distributed agreement, where synchronization depends on communication, here dependence is both ways. Therefore, from a theoretical perspective our approach can be viewed as a way to jointly control communication and synchronization. We finally illustrate our approach in computer simulations.**

I. Introduction

Most tasks in the field of multi-robot systems critically rely on frequent communication between the members of the team. Nevertheless, environmental parameters or system specifications, such as noise and signal strength, respectively, often interfere with the required communication conditions. Unlike most related work that assumes or enforces the necessary communication conditions and focuses on completing a team objective, we propose a distributed coordination scheme for very-low-range (VLR) wireless agents that can only communicate when they are physically very close to each other. Among others, one important application of this framework is in the area of underwater robotics where communication between distant agents is typically difficult.

As in most applications involving multi-robot coordination, connectivity of the underlying communication network is required here too to achieve the desired task. In the literature so far, connectivity has been either considered an assumption as in the case of consensus [1]–[3], or has been treated as an optimization [4]–[7] or feedback control objective [8]–[12], respectively. Common in these approaches are disc-based communication models that perform poorly when interference is present. Furthermore, they enforce connectivity for all time, which significantly restricts the range of tasks that can be achieved with VLR wireless agents that need to occasionally disconnect from the team to pursue their individual objectives. Coordinated control in the presence of interference is captured more accurately by i.i.d. [13], [14] or time-dependent [15], [16] random networks, which also allow occasional disconnection of the networks. Nevertheless, in these approaches, communication is no longer a control objective.

In this paper, we synchronize a team of VLR wireless agents at a set of predefined rendezvous or communication points. We call this problem "Synchronous Rendezvous", with "synchronous" and "rendezvous" referring to temporal and spacial synchronization, respectively. The agents travel back and forth between two rendezvous points and employ distributed consensus to agree on their travel times and waiting times at the rendezvous points. Under the assumption that the set of rendezvous points and agent paths forms a bipartite network with a 2-coloring of its nodes, connectivity of the dynamic communication network is almost always eventually guaranteed and, therefore, periodic and simultaneous arrivals at same color rendezvous points (synchronous rendezvous) is achieved. Our approach relies on the observation that this problem is essentially equivalent to the problem of synchronizing a network of harmonic oscillators that can only communicate when they meet at the north or south pole of their circular trajectories. The resulting multi-agent system integrates continuous-time agent motion with discrete-time coordination at the rendezvous points and is, therefore, hybrid. Furthermore, the feedback nature of our approach makes it robust to disturbances that are often present, for instance, in underwater environments with strong currents.

Unlike existing work on consensus, where synchronization depends on communication [1]–[3], [10], [13]–[16], here dependence goes both ways. This work is related to [17], [18], where a group of agents are synchronized on a one dimensional ring or segment to perform perimeter surveillance tasks. However, moving from rings to networks with higher node degrees, adds not only complexity to the problem, but also a broader range of possible applications, such as exploration or surveillance of three dimensional spaces. Unlike [17], [18] where the rendezvous points on the ring can be determined dynamically, in arbitrary networks as we discuss here, they need to be specified or decided a priori. Nevertheless, the rendezvous time instants are still the result of distributed consensus, hence the feedback nature of our approach.

The rest of this paper is organized as follows. In Section II we define the problem under consideration and develop the necessary graph theoretic background to address it. In Section III we define the hybrid agent and develop the distributed control protocols for synchronization, while in Section IV we show correctness of our approach, namely, periodic connectivity and, therefore, synchronous rendezvous. Computer simulations are presented in Section V.

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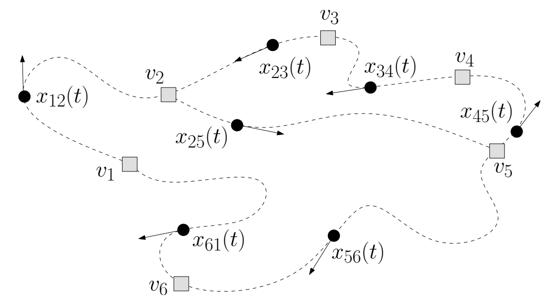


Fig. 1. Path network $\mathcal{G}_P$ embedded in $\mathbb{R}^2$ with $n = 7$ agents traveling along its links (dashed lines). Rendezvous points are denoted with squares, while agents are denoted with circles.

II. Problem Definition

Consider a set of $r > 0$ distinct *rendezvous* points in $\mathbb{R}^d$ and denote by $v_i \in \mathbb{R}^d$ the coordinates of rendezvous point i. Let also $\gamma_{ij} : [0, 1] \to \mathbb{R}^d$ denote a smooth curve, or *path*, joining rendezvous points i and j such that $v_i = \gamma_{ij}(0)$ and $v_j = \gamma_{ij}(1)$. The union of rendezvous points and paths gives rise to a *path* network $\mathcal{G}_P = (\mathcal{V}_P, \mathcal{E}_P)$ consisting of a set $\mathcal{V}_P = \{1, \dots, r\}$ of nodes indexed by the set of rendezvous points and a set of links $\mathcal{E}_P \subseteq \mathcal{V}_P \times \mathcal{V}_P$ associated with paths between rendezvous points such that $(i, j) \in \mathcal{E}_P$ if and only if the path γ_{ij} exists. We assume that $\mathcal{G}_P$ is an undirected network, i.e., $(i, j) \in \mathcal{E}_P$ whenever $(i, j) \in \mathcal{E}_P$. Any pair of nodes $i, j \in \mathcal{V}_P$ that are joined by a link $(i, j) \in \mathcal{E}_P$ are called adjacent or neighbors in $\mathcal{G}_P$. Hence, we can define the set of neighbors of node i by $\mathcal{N}_P^i \triangleq \{j \in \mathcal{V}_P : (i, j) \in \mathcal{E}_P\}$.

Consider further a group of $n = |\mathcal{E}_P|$ agents associated with the links in $\mathcal{E}_P$ and let $x_{ij}(t) \in \mathbb{R}^d$ denote the position of agent a_{ij} at time $t > 0$.[1] We assume that every agent a_{ij} can travel along the curve γ_{ij} with velocity (Fig. 1)

$$\dot{x}_{ij}(t) = u_{ij}(t) \in T_{x_{ij}(t)}\left(\gamma_{ij} \backslash \partial\gamma_{ij}\right),$$

where $T_{x_{ij}(t)}\left(\gamma_{ij} \backslash \partial\gamma_{ij}\right)$ is the tangent space of $(\gamma_{ij} \backslash \partial\gamma_{ij})$ at $x_{ij}(t)$ and $\partial\gamma_{ij} = \{v_i, v_j\}$ denotes the boundary of the curve γ_{ij}. Also, we assume that the agents can only communicate with each other when they physically meet at common rendezvous points. We call such agents, very-low-range (VLR) wireless agents. Therefore, we can define a dynamic *agent* communication network $\mathcal{G}_A(t) = (\mathcal{V}_A, \mathcal{E}_A(t))$, where $\mathcal{V}_A = \{1, \dots, n\}$ is the set of nodes indexed by the set of agents and $\mathcal{E}_A(t) \subseteq \mathcal{V}_A \times \mathcal{V}_A$ is the set of communication links such that $(a_{ij}, a_{ik}) \in \mathcal{E}_A(t)$ if and only if $x_{ij}(t) = x_{ik}(t) = v_i$, where $i \in \mathcal{V}_P$. Similarly, we can define the set of neighbors of agent a_{ij} at time $t > 0$ by $\mathcal{N}_A^{ij}(t) \triangleq \{a_{ik} \in \mathcal{V}_A : (a_{ij}, a_{ik}) \in \mathcal{E}_A(t)\}$. Then, the problem addressed in this paper can be stated as follows:

[1]Since $\mathcal{G}_P$ is undirected, the order of the indices in a_{ij} or $x_{ij}(t)$ does not matter. Also, $|\mathcal{E}_P|$ indicates the cardinality of the set $\mathcal{E}_P$.

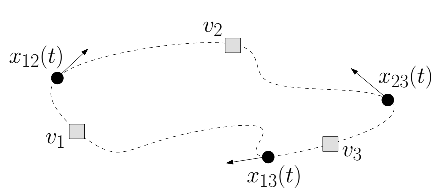


Fig. 2. Path network on three nodes, on which it is impossible to achieve simultaneous rendezvous of the agents at the rendezvous points so that they are all full or empty.

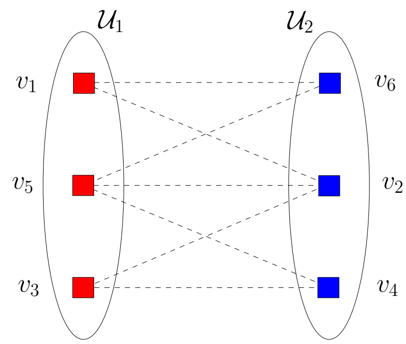


Fig. 3. Partition of the nodes of the bipartite network in Fig. 1 into two disjoint sets $\mathcal{U}_1$ and $\mathcal{U}_2$ containing red and blue nodes, respectively. Periodic synchronization in Definition 2.1 is equivalent to all agents periodically switching between the sets $\mathcal{U}_1$ and $\mathcal{U}_2$.

Problem 1: For any initialization $(x_{ij}(0), u_{ij}(0))$ of n agents on a connected bipartite path network $\mathcal{G}_P$, derive distributed controllers $u_{ij}(t)$ that *synchronize rendezvous* of the agents at the rendezvous points $\mathcal{V}_P$.

In the rest of this section we define the notion of synchronous rendezvous and show that *connectivity* and *bipartiteness* of the path network are necessary conditions for synchronization. In particular, we have the following definition:

Definition 2.1 (Synchronous rendezvous): Let $n_i(t) \triangleq \{a_{ij} \in \mathcal{V}_A : x_{ij}(t) = v_i\}$ denote the number of agents that are at rendezvous point $i \in \mathcal{V}_P$ at time $t > 0$. We say that a set of n agents rendezvous synchronously on a path network $\mathcal{G}_P$ if for all $i \in \mathcal{V}_P$:

$$n_i(t) \in \begin{cases} \{0, |\mathcal{N}_P^i|\}, & \text{if } t = \tau, 2\tau, 3\tau, \dots \\ \{0\}, & \text{otherwise} \end{cases},$$

where $\tau > 0$ denotes the period of synchronization.

In other words, Definition 2.1 implies that all agents simultaneously and periodically visit the rendezvous points and when this happens, every rendezvous point is either *empty* (no agents present) or *full* (all agents present). Clearly, this notion of synchronization is not possible for arbitrary path networks as shown in Fig. 2. However, it is possible for bipartite path networks (Fig. 3) as we show next [19]:

Definition 2.2 (Bipartite graphs): A graph is called bipartite if its nodes can be divided into two disjoint sets $\mathcal{U}_1$ and $\mathcal{U}_2$ such that every link connects a node in $\mathcal{U}_1$ to a node in $\mathcal{U}_2$. Equivalently, a bipartite graph is a graph that does not contain any odd-length cycles, i.e., odd numbers of nodes connected in closed chains.

It is easy to see that the path network being bipartite is a necessary condition for synchronous rendezvous. For this we employ the following definition of graph coloring [19]:

Definition 2.3 (Graph coloring): Graph coloring is the labeling of a graph's nodes with colors so that no two nodes

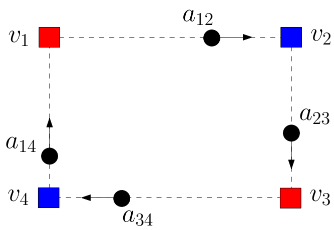


Fig. 4. Four agents traveling along the links of a cyclic path network on four nodes. For unbounded waiting times at all nodes and clockwise initialization of the agent velocities, all agents a_{14}, a_{12}, a_{23} and a_{34} will end up at nodes 1, 2, 3 and 4, respectively, waiting indefinitely for a neighbor to arrive. Therefore, the system will deadlock. However, for bounded waiting times at the red nodes, agents a_{14} and a_{23} will eventually leave and meet with agents a_{34} and a_{12}, respectively, that have been waiting at the blue nodes 2 and 4.

sharing the same link have the same color. A coloring using at most k colors is called a k-coloring.

A graph that can be assigned a k-coloring is called k-colorable. A subset of nodes assigned to the same color is called a color class, and every such class forms an independent set. Therefore, a k-coloring is equivalent to a partition of the node set into k independent sets, which implies that the terms k-partite and k-colorable are equivalent. It follows that the only 1-colorable graphs are the linkless graphs, while the only 2-colorable graphs are the bipartite graphs including trees and forests. Since all agents can only rendezvous with other agents at either one of their two adjacent rendezvous points in $\mathcal{G}_P$, synchronous rendezvous can be equivalently though of as simultaneous and periodic rendezvous of all agents at the same color of a 2-colorable graph (Fig. 3). Therefore, a necessary condition for synchronous rendezvous is that the path network is 2-colorable, or bipartite.

Finally, as in all literature on synchronization and consensus [1]–[3], connectivity of the agent network $\mathcal{G}_A(t)$ over time is necessary here too.[2] However, in our framework connectivity can only be periodic due to periodic communication at the rendezvous points and depends on both the distributed controllers and the structure of the path network. In the following section we show that as long as the path network $\mathcal{G}_P$ is bipartite and connected such controllers exist and we are able to explicitly characterize them.

III. Distributed Coordination

As discussed in Section II, let $\mathcal{G}_P$ denote a bipartite path network and assume a 2-coloring of its nodes as in Definition 2.3. Assume that $\mathcal{G}_P$ is embedded in $\mathbb{R}^d$ and that the paths γ_{ij} between adjacent rendezvous points $i, j \in \mathcal{V}_P$ are straight lines. Consider finally n agents traveling back and forth along the links of $\mathcal{G}_P$ and let $2\tau_{ij} > 0$ denote the *period* of motion of agent a_{ij}, i.e., the time required for agent a_{ij} to travel from point i to point j and back. For simplicity, we assume constant agent speed so that

$$\dot{x}_{ij} = \frac{(-1)^{I[j\to i]}}{\tau_{ij}} \frac{v_i - v_j}{\|v_i - v_j\|_2}, \tag{1}$$

where $I[j \to i] \in \{0, 1\}$ is an indicator function such that

$$I[j \to i] = \begin{cases} 1, & \text{if direction of motion is from } j \text{ to } i \\ 0, & \text{otherwise} \end{cases}.$$

Since coordination only takes place when agents meet at a rendezvous point, we introduce a waiting time $w_{ij} > 0$ that starts ticking once agent a_{ij} has arrived to one of its two adjacent rendezvous points i or j. If the waiting times can be unbounded, i.e., if the agents can wait at the rendezvous points until all neighbors have arrived and the nodes are full, then the system could easily deadlock for certain initializations of the agent velocities, as shown in Fig. 4. To address this challenge, we assume bounded waiting times at the *red* rendezvous points and unbounded waiting times at the *blue* ones. Then, deadlocks as the one shown in Fig. 4 can be avoided, since agents at red nodes will eventually leave and meet with their neighbors at the blue nodes. Besides resolving deadlocks, this construction also ensures connectivity at the blue nodes, i.e., information propagation along every direction/link adjacent to these nodes. Clearly, connectivity at the red nodes *infinitely often* is then sufficient for connectivity of the agent network $\mathcal{G}_A(t)$ over time. In view of this observation, we restrict communication to connected, namely, full nodes. In other words, even though agents might be waiting at red nodes, communication does not take place unless these nodes are full.

A. The Hybrid Agent

With the above terminology, we can now propose a hybrid controller for agent a_{ij} (Fig. 5). In particular, for any initialization $(x_{ij}(0), \dot{x}_{ij}(0), \tau_{ij}(0))$ of agent a_{ij} on the link $(i, j) \in \mathcal{E}_P$, the agent enters a "Travel" mode where it moves towards either one of its red or blue adjacent rendezvous points with constant speed according to equation (1).[3] Once agent a_{ij} has arrived at a rendezvous point, then if this node is red, i.e., if $x_{ij}(t) = v_j$, it enters a "Red" mode where it waits until either the node is full, i.e., until $n_j(t) = |\mathcal{N}_P^j|$, or until it has waited for longer than a fraction of its period, i.e., until $w_{ij}(t) = \delta\tau_{ij}(t)$ with $0 < \delta < 1$. If the red node becomes full before the waiting time reaches its threshold, then all neighbors communicate their periods and average their values according to the following discrete-time consensus update

$$\tau_{ij}(t) = \frac{1}{|\mathcal{N}_P^j(t)| + 1} \sum_{k \in \mathcal{N}_P^j} \tau_{kj}(t). \tag{2}$$

and enters again the "Travel" mode, this time moving towards the blue rendezvous point.[4] On the other hand, if the waiting time reaches its threshold before the red node is full, then agent a_{ij} leaves the node without performing any updates and enters a "Fast Travel" mode where it moves fast towards the blue rendezvous point to make up for the

[2] We say that a network is connected if there exists a sequence of distinct adjacent nodes (a chain) between any two of its nodes.

[3] Without loss of generality, assume that node i is blue and node j is red.

[4] Note that the objective of the averaging rule (2) is that all agents eventually have the same period and rendezvous simultaneously at the nodes of $\mathcal{G}_P$.

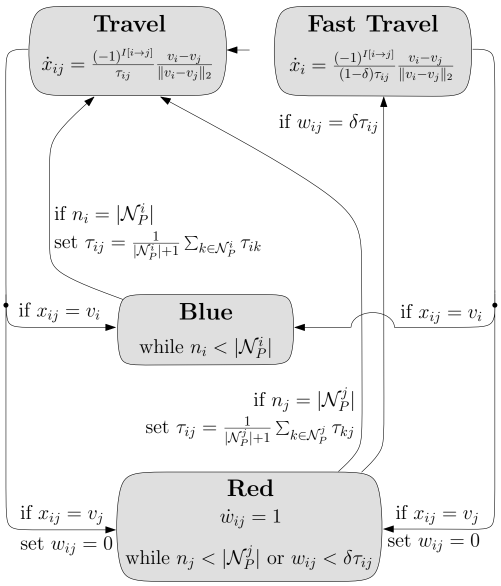


Fig. 5. Hybrid automaton of agent a_{ij} traveling along the link $(i, j) \in \mathcal{E}_P$. The rendezvous points i and j correspond to blue and red nodes, respectively.

time spent while waiting. Clearly, the larger δ is the more likely it is that the red node will eventually become full and communication will be established, but the less time is left to travel back to the blue node if connectivity at the red node fails. Therefore, there is a tradeoff involved in the choice of δ, which is typically chosen to satisfy the maximum velocity specifications of the agents. As before, agent a_{ij} remains in the "Travel" or "Fast Travel" mode until it reaches the blue rendezvous point i, i.e., until $x_{ij}(t) = v_i$, when it enters the "Blue" mode. Once in the "Blue" mode, agent a_{ij} waits indefinitely until this node is full, i.e., until $n_i(t) = |\mathcal{N}_P^i|$, and when this happens, it averages its period with its neighbors as in equation (2) and jumps back to the "Travel" mode as before (Fig. 5).

IV. Correctness of the Overall System

Let $\mathbb{A} = \times_{a_{ij}\in\mathcal{V}_A} \mathbb{A}_{ij}$ denote the composition (or product) of all individual agents $\mathbb{A}_{ij}$ which captures the properties of the overall system. To show that the system $\mathbb{A}$ of agents will eventually achieve synchronous rendezvous on the nodes of the bipartite network $\mathcal{G}_P$ we need to show that eventually all rendezvous points are either full or empty, or equivalently that agents arrive simultaneously at the rendezvous points. Our result relies on the observation that our construction is essentially equivalent to a system of n harmonic oscillators each one described by

$$y_{ij}(t) = \sin\left(\frac{2\pi}{p_{ij}} t + \theta_{ij}\right), \tag{3}$$

where $p_{ij} = 2\tau_{ij}$ is the period of oscillation and $\theta_{ij} = \left(\frac{2\pi}{p_{ij}} w_{ij}\right) \mathrm{mod} 2\pi$ is its phase. Then, the position of agent a_{ij} in $\mathbb{R}^d$ is given by the convex combination

$$x_{ij}(t) = \frac{y_{ij}(t)+1}{2} v_i + \left(1 - \frac{y_{ij}(t)+1}{2}\right) v_j, \tag{4}$$

which can also be normalized to satisfy the constant speed assumption suggested in (1).[5] Therefore, synchronous rendezvous of the n agents is equivalent to synchronization of the n harmonic oscillators described by (3), which can be achieved only if the agent network $\mathcal{G}_A(t)$ is connected infinitely often. Synchronization of oscillators has been extensively studied when communication between adjacent oscillators does not depend on their states, i.e., their period and phase of oscillation [20] (see also literature on consensus [1]–[3], [10], [13]–[16]). However, in our case communication is state dependent and can only take place when the agents meet at a rendezvous point. Since, all blue nodes are connected infinitely often by construction, we only need to show that the red nodes are also connected over time. In particular, we have the following result.

Proposition 4.1 (Connectivity of the red nodes): If there do not exist periods τ_{ij} that are integer multiples of each other, then there always exists a $0 < \Delta < 1$ such that if $\delta > \Delta$, then the red rendezvous points are connected infinitely often. If periods that are integer multiples of each other exist, then connectivity of the red nodes depends on initialization.

Proof: Assume that no rendezvous at a red node has taken place yet. Then, $\mathcal{G}_A(t)$ consists of at least $|\mathcal{V}_P|/2$ connected components, associated with connected blue nodes and isolated agents, and clearly no information can propagate in $\mathcal{G}_A(t)$. Hence, the waiting times w_{ij} and periods τ_{ij} of all agents a_{ij} remain constant.[6] Let j be the first red node where the agents a_{ij} with $i \in \mathcal{N}_P^j$ rendezvous and denote by $t_{ij} > 0$ the sequence of time instants that agent a_{ij} arrives at node j, given by

$$\sin\left(\frac{2\pi}{2\tau_{ij}}(t_{ij} + w_{ij})\right) = 1 \quad \Rightarrow \quad t_{ij} = 2\tau_{ij} m_{ij} + \frac{1}{2}\tau_{ij} - w_{ij},$$

for all $m_{ij} = 0, 1, 2, \ldots$. Then, the agents rendezvous at node j if and only if $|t_{ij} - t_{kj}| < \delta\tau_{ij}$ for all $k \in \mathcal{N}_P^j$, where without loss of generality it is assumed that agent a_{ij} has arrived first. Substituting the arrival times t_{ij} in the rendezvous condition above we get

$$\left| m_{ij} - \frac{\tau_{kj}}{\tau_{ij}} m_{kj} + \underbrace{\left(\frac{1}{4} - \frac{w_{ij}}{2\tau_{ij}} - \frac{\tau_{kj}}{4\tau_{ij}} + \frac{w_{kj}}{2\tau_{ij}}\right)}_{\beta_{ik}} \right| < \frac{\delta}{2},$$

or equivalently

$$\left| m_{ij} - \left\lfloor \frac{\tau_{kj}}{\tau_{ij}} m_{kj} \right\rfloor - \left(\frac{\tau_{kj}}{\tau_{ij}} m_{kj}\right) \mathrm{mod} 1 \right. \\ \left. + \lfloor \beta_{ik} \rfloor + \beta_{ik} \mathrm{mod} 1 \right| < \frac{\delta}{2}, \tag{5}$$

[5] Note that coordination depends only on travel times and not on the specific motion patterns. In other words, our results hold for any motion schemes that satisfy the given travel times.

[6] Note that after waiting for time $\delta\tau_{ij}$ at a red node, agent a_{ij} returns to its adjacent blue node in time $(1-\delta)\tau_{ij}$ (Fig. 5). Also, after the first rendezvous at a blue node, all involved agents share the same information and, therefore, keep arriving at this node simultaneously, until a rendezvous at an adjacent red node has taken place.

where $\lfloor x \rfloor$ and $x\text{mod}1$ denote the floor and residual of the real number $x \in \mathbb{R}$, respectively, i.e., $x = \lfloor x \rfloor + x\text{mod}1$. Since $w_{ij} < 2\tau_{ij}$, we have that $-\frac{3}{4} < \frac{1}{4} - \frac{w_{ij}}{2\tau_{ij}} < \frac{1}{4}$ and $\frac{w_{kj}}{2\tau_{ij}} - \frac{\tau_{kj}}{4\tau_{ij}} \leq \frac{3\tau_{kj}}{4\tau_{ij}}$. Hence,

$$\begin{aligned}\lfloor \beta_{ik} \rfloor &\leq \left\lfloor \frac{1}{4} - \frac{w_{ij}}{2\tau_{ij}} \right\rfloor + \left\lfloor \frac{w_{kj}}{2\tau_{ij}} - \frac{\tau_{kj}}{4\tau_{ij}} \right\rfloor + 1 < \left\lfloor \frac{3\tau_{kj}}{4\tau_{ij}} \right\rfloor + 1 \\ &\leq \left\lfloor \frac{3\max_{a_{ij}\in\mathcal{V}_A}\{\tau_{ij}\}}{4\min_{a_{ij}\in\mathcal{V}_A}\{\tau_{ij}\}} \right\rfloor + 1,\end{aligned}$$

i.e., $\lfloor \beta_{ik} \rfloor$ is upper bounded as long as $\min_{i,j\in\mathcal{V}_P}\{\tau_{ij}\} > 0$, which is guaranteed due to boundedness of the agent velocities. Therefore, there exist finite integers $m_{ij}, m_{kj} \geq 0$ such that

$$m_{ij} = \left\lfloor \frac{\tau_{kj}}{\tau_{ij}} m_{kj} \right\rfloor - \lfloor \beta_{ik} \rfloor . \quad (6)$$

Substituting equation (6) in (5) we get

$$\left| \left(\frac{\tau_{kj}}{\tau_{ij}} m \right) \text{mod}1 - \beta_{ik}\text{mod}1 \right| < \frac{\delta}{2}, \quad (7)$$

which essentially represents a set of intervals of length $\delta > 0$ centered at $0 < \left\{ \left(\frac{\tau_{kj}}{\tau_{ij}} m \right) \text{mod}1 \right\}_{m=0}^{\infty} < 1$.[7] To show that the agents a_{ij} with $i \in \mathcal{N}_P^j$ will eventually rendezvous at the red node j we need to show that there exist an integer m for which equation (7) is satisfied for $0 < \delta < 1$.

Assuming that the periods $\tau_{ij}, \tau_{kj} > 0$ are rational numbers,[8] there exist positive integers $\mu_{ik}, \nu_{ik} > 0$ such that

$$\frac{\tau_{kj}}{\tau_{ij}} = \frac{\mu_{ik}}{\nu_{ik}},$$

where ν_{ik} is taken to be the smallest such integer. Then, the sequence $\left\{ \left(\frac{\tau_{kj}}{\tau_{ij}} m \right) \text{mod}1 \right\}_{m=0}^{\infty}$ essentially splits consecutive intervals of length μ_{ik}, each one defined by end points obtained for $m = q\nu_{ik}$ and $m = (q+1)\nu_{ik}$ with $q = 0, 1, 2, \ldots$, into ν_{ik} equal subintervals of length $\frac{\mu_{ik}}{\nu_{ik}}$ each. This implies that there are exactly $\nu_{ik} - 1$ distinct nonzero residuals $\left(\frac{\tau_{kj}}{\tau_{ij}} m \right) \text{mod}1$, which repeat themselves periodically. Therefore, the quantity $\beta_{ik}\text{mod}1$ lies in one of the $\nu_{ik} - 1$ distinct intervals of length δ defined in (7) if their union covers the whole unit interval [0,1], i.e., if

$$\frac{\delta}{2} > \frac{1}{2} \max_{l \leq \nu_{ik}-1} \left| \left(\frac{\tau_{kj}}{\tau_{ij}} m_l \right) \text{mod}1 - \left(\frac{\tau_{kj}}{\tau_{ij}} m_{l+1} \right) \text{mod}1 \right|, \quad (8)$$

where the index $l > 0$ is such that

$$m_{l+1} = \min_{m \leq \nu_{ik}-1} \left\{ \left(\frac{\tau_{kj}}{\tau_{ij}} m \right) \text{mod}1 > \left(\frac{\tau_{kj}}{\tau_{ij}} m_l \right) \text{mod}1 \right\},$$

i.e., such that m_l and m_{l+1} correspond to consecutive residuals. Condition (8) should hold for all agents a_{kj} with $k \in \mathcal{N}_P^j$, which gives

$$\Delta = \max_{\substack{l \leq \nu_{ik}-1 \\ i,k\in\mathcal{N}_P^j}} \left| \left(\frac{\tau_{kj}}{\tau_{ij}} m_l \right) \text{mod}1 - \left(\frac{\tau_{kj}}{\tau_{ij}} m_{l+1} \right) \text{mod}1 \right| < 1.$$

Suppose now that there exist periods τ_{ij} and τ_{kj} that are multiples of each other. Then, $\nu_{ik} = 1$ which gives $\Delta = 1$, i.e., connectivity of the red node j can not be guaranteed. Nevertheless, letting $\tau_{kj} = \mu_{ik}\tau_{ij}$ with $\mu_{ik} = 1, 2, \ldots$ in equation (5), we get

$$\left| m_{ij} - \mu_{ik} m_{kj} + \frac{1}{4} - \frac{w_{ij}}{2\tau_{ij}} - \frac{\mu_{ik}}{4} + \frac{w_{kj}}{2\tau_{ij}} \right| < \frac{\delta}{2},$$

or equivalently

$$\left| \left(\frac{\mu_{ik} - 1}{4} \right) \text{mod}1 - \left(\frac{w_{ij} - w_{kj}}{2\tau_{ij}} \right) \text{mod}1 \right| < \frac{\delta}{2}.$$

Since $\left(\frac{\mu_{ik}-1}{4} \right) \text{mod}1 \in \left\{ \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \right\}$ we conclude that $0 < \delta < 1$ still exists as long as $\frac{1}{4} < \left(\frac{w_{ij}-w_{kj}}{2\tau_{ij}} \right) \text{mod}1 < \frac{3}{4}$. In other words connectivity of the red nodes depends on the initial phases of oscillation. Note that the above arguments can be applied iteratively to ensure that the red nodes are connected infinitely often. Note also that at the equilibrium, i.e., when $\tau_{kj} = \tau_{ij}$ ($\mu_{ik} = 1$) and $w_{kj} = w_{ij}$, setting $\delta \to 0$ is sufficient for synchronization, as desired. ■

Proposition 4.1 together with the fact that the blue nodes are connected infinitely often by construction, implies that the agent network $\mathcal{G}_A(t)$ is connected infinitely often too. Hence, we have the following result.

Proposition 4.2 (Synchronous rendezvous): If the conditions of Proposition 4.1 are satisfied, then synchronous rendezvous of all agents on the nodes of $\mathcal{G}_P$ is guaranteed.

Proof: This result relies on the observation that updating of the period and phase of the oscillators defined in equation (3) is according to the following *average* and *minimum* consensus rules[9]

$$\tau_{ij} = \frac{1}{|\mathcal{N}_P^j| + 1} \sum_{k\in\mathcal{N}_P^j} \tau_{kj} \quad (9)$$

and

$$w_{ij} = \min_{k\in\mathcal{N}_P^j} \{w_{kj}\}. \quad (10)$$

If the conditions of Proposition 4.1 are satisfied, then the agent network $\mathcal{G}_A(t)$ is connected infinitely often and the periods and phases of all agents will eventually become equal (cf. [1]). Therefore, synchronous rendezvous will be achieved, since by (4) all agents will be simultaneously switching between their adjacent red and blue nodes. ■

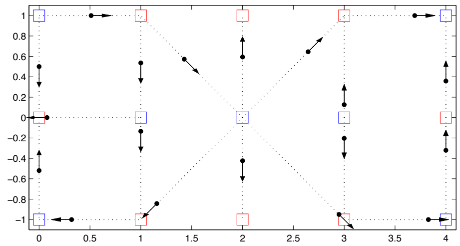
V. Simulations

In this section we illustrate our approach on various bipartite path networks embedded in $\mathbb{R}^2$, for random initialization of the positions x_{ij}, periods τ_{ij} and orientations of the agents on their assigned paths. As discussed above, we assume straight line paths and constant velocities that depend on the periods of motion according to equation (1). In all cases and for all agents, the waiting time is chosen to be 30% of the period of motion τ_{ij}, i.e., $\delta = .3$.

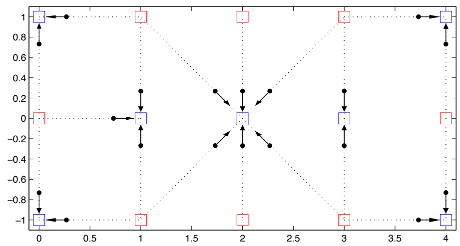
[7] To simplify notation we have dropped dependence of m on indices.

[8] If initialized rational, equation (2) ensures that all periods remain rational for all time.

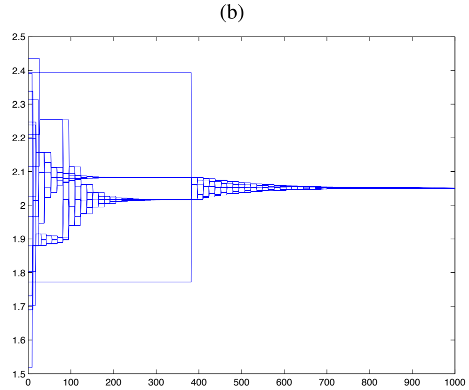
[9] Note that agents with small phases arrive late at rendezvous points and do not need to wait for long. Since the agents leave a rendezvous point as soon as the last agent arrives, synchronization is with respect to the agent with smallest phase, hence, the minimum consensus rule.



(a)



(b)



(c)

Fig. 6. Synchronous rendezvous of 19 agents traveling along the links of an underlying bipartite path network. Fig. 6(a) initial configurations, Fig. 6(b) rendezvous at the blue nodes, Fig. 6(c) consensus of travel times (periods). Note the idle time intervals during which certain rendezvous points are disconnected.

Fig. 6 shows synchronous rendezvous of 19 agents traveling along the links of a bipartite path network with irregular node degrees. This shows that the proposed synchronization scheme depends only on bipartiteness and not on the topology of the path network. Figs. 6(a) and 6(b) show initial and asymptotic configurations of the agents, while Fig. 6(c) illustrates consensus of the periods τ_{ij} of the agents. While certain red nodes are disconnected over periods of time, connectivity is eventually reestablished and consensus is achieved, as expected by Propositions 4.1 and 4.2.

VI. Conclusions

In this paper, we addressed the problem of synchronous rendezvous of a team of mobile agents on the nodes of a bipartite network. This problem has many applications in scenarios where the communication range of the agents is much smaller compared to the size of the configuration space where the agents need to operate. We called such agents, very-low-range (VLR) wireless agents and proposed a distributed coordination algorithm that allowed them to split in order to complete their individual tasks and periodically rendezvous at given communication points in order to exchange information. Unlike prior work on distributed agreement, where synchronization depends on communication, here dependence goes both ways. Therefore, our approach can be viewed as a way to jointly control communication and synchronization.

References

[1] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, June 2003.

[2] W. Ren and R. W. Beard, "Consensus seeking in multi-agent systems under dynamically changing interaction topologies," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 655–661, May 2005.

[3] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. of the IEEE*, vol. 95, no. 1, pp. 215–233, January 2007.

[4] Y. Kim and M. Mesbahi, "On maximizing the second smallest eigenvalue of a state-dependent graph laplacian," *IEEE Transactions on Automatic Control*, vol. 51, no. 1, pp. 116–120, January 2006.

[5] M. C. DeGennaro and A. Jadbabaie, "Decentralized control of connectivity for multi-agent systems," in *Proc. 45th IEEE Conference on Decision and Control*, San Diego, CA, December 2006, pp. 3628–3633.

[6] S. Boyd, "Convex optimization of graph laplacian eigenvalues," in *Proc. International Congress of Mathematicians*, Madrid, Spain, August 2006, pp. 1311–1319.

[7] K. Srivastava and M. W. Spong, "Multi-agent coordination under connectivity constraints," in *Proc. 2008 American Control Conference*, Seattle, WA, June 2008, pp. 2648–2653.

[8] M. M. Zavlanos and G. J. Pappas, "Distributed connectivity control of mobile networks," *IEEE Transactions on Robotics*, vol. 24, no. 6, pp. 1416–1428, December 2008.

[9] ——, "Potential fields for maintaining connectivity of mobile networks," *IEEE Transactions on Robotics*, vol. 23, no. 4, pp. 812–816, August 2007.

[10] M. Ji and M. Egerstedt, "Coordination control of multi-agent systems while preserving connectedness," *IEEE Transactions on Robotics*, vol. 23, no. 4, pp. 693–703, August 2007.

[11] M. Schuresko and J. Cortes, "Distributed motion constraints for algebraic connectivity of robotic networks," *Journal of Intelligent and Robotic Systems*, vol. 56, no. 1-2, pp. 99–126, September 2009.

[12] G. Notarstefano, K. Savla, F. Bullo, and A. Jadbabaie, "Maintaining limited-range connectivity among second-order agents," in *Proc. 2006 American Control Conference*, Minneapolis, MN, June 2006, pp. 2124–2129.

[13] A. Tahbaz-Salehi and A. Jadbabaie, "A necessary and sufficient condition for consensus over random networks," *IEEE Transactions on Automatic Control*, vol. 53, no. 3, pp. 791–795, April 2008.

[14] Y. Hatano and M. Mesbahi, "Agreement over random networks," *IEEE Transactions on Automatic Control*, vol. 50, no. 11, pp. 1867–1872, November 2005.

[15] A. Tahbaz-Salehi and A. Jadbabaie, "Consensus over ergodic stationary graph processes," *IEEE Transactions on Automatic Control*, vol. 55, no. 1, pp. 225–230, January 2010.

[16] G. Picci and T. Taylor, "Almost sure convergence of random gossip algorithms," in *Proc. 46th IEEE Conference on Decision Control*, New Orleans, LA, December 2007, pp. 282–287.

[17] S. Susca, F. Bullo, and S. Martinez, "Synchronization of beads on a ring," in *Proc. 46th IEEE Conf. on Decision and Control*, New Orleans, LA, December 2007, pp. 4845–4850.

[18] H. Wang and Y. Guo, "Synchronization on a segment without localization: Algorithm and applications," in *Proc. IEEE/RSJ International Conference on Intelligent Robots and Systems*, St. Louis, MO, October 2009, pp. 3441–3446.

[19] C. Godsil and G. Royle, *Algebraic Graph Theory*, ser. Graduate Texts in Mathematics. Berlin, Germany: Springer-Verlag, 2001, vol. 207.

[20] A. Jadbabaie, N. Motee, and M. Barahona, "On the stability of the kuramoto model of coupled nonlinear oscillators," in *Proc. American Control Conference*, Boston, MA, June 2004, pp. 4296–4301.