Distributed Continuous-time Online Optimization Using Saddle-Point Methods

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***Abstract*— This paper introduces saddle-point methods for distributed continuous-time online convex optimization, where the system objective function varies arbitrarily over time subject to some global inequality constraints. The overall dynamics of the proposed saddle-point controller are described by a system of differential equations, coupled linearly through the network Laplacian. The controller pushes actions along the negative gradient direction of the objective, constraint violation, as well as network disagreement using only causal and locally available information, while dynamically adapting the Lagrange multipliers in a decentralized fashion. We define *regret* as the cost difference with the optimal action over time. We show that the proposed saddle-point controller achieves a regret of order $O(\sqrt{T})$ with the time horizon T. We also address the impact of the network topology, encoded in the spectrum of the network Laplacian, as a factor on the speed of convergence.**

I. INTRODUCTION

Decentralized optimization problems defined over multi-agent networks are ubiquitous in control, machine learning, signal processing, and other areas in science and engineering (see e.g. [1]–[4]). This necessitates the development of new algorithms that can properly handle various forms of decentralized optimization problems appearing in these applications, including the ones where the network agents are coupled through decision variables, objective functions, and/or constraints.

The method of saddle point, originally developed by Arrow-Hurwicz-Uzawa [5] in the late 1950's, has been recently revisited in the field of decentralized optimization [6]–[10]. The Saddle Point Method (SPM) is a primal-dual method which can efficiently handle equality and/or inequality constraints of an optimization problem by alternating between updates of decision variables (primal variables) and Lagrange multipliers (dual variables). The SPM is especially useful in decentralized optimization if the constraints are explicitly given in the form of inequality constraints and the network agents are coupled through decision variables.

In this paper, we study an online continuous-time variant of the SPM for solving online convex optimization problems with separable objective functions, whose components are distributed over the agents of a network, and global inequality constraints. Our algorithm builds on the recently proposed online discrete-time distributed SPMs [9], [10] and online continuous-time centralized SPM [11], which we modify to use the network Laplacian as a means of obtaining a network-wide agreement. The main purpose of the development of this algorithm is to properly handle the continuously time-varying objectives $\sum_{i=1}^{N} f_0^i(t,\mathbf{x})$ and global constraints of the form $f(t,\mathbf{x}) \preceq \mathbf{0}$, where $\mathbf{x}$ is the network-wide decision variables shared among all agents and $f_0^i(t,\cdot)$ is only known to agent i. We show that our developed algorithm obtains a sublinear regret over a connected undirected graph.

Continuous-time distributed SPM has advantages in its own right in the context of distributed control systems whenever signals are inherently continuous quantities. Note that the behavior of many discrete-time approaches can be characterized by studying continuous-time dynamics first as the discretization of which naturally provides practical algorithms. We also mention that the discretized version of the continuous algorithm that we propose in this paper is different from the approaches studied in [9], [10].

II. PRELIMINARIES

In this section, we introduce basic Lagrangian duality, saddle point theorem and the original Arrow-Hurwicz-Uzawa [5] method.

Consider the following constrained optimization problem:

$$\min f_0(\mathbf{x}) \quad \text{s.t. } f(\mathbf{x}) \preceq \mathbf{0},\ \mathbf{x} \in X, \tag{1}$$

where $f = [f_1,\ldots,f_m]^\top$, $f_j : \mathbb{R}^n \to \mathbb{R}$ for $j = 0,\ldots,m$ are convex functions, and $X \subseteq \mathbb{R}^n$ is a nonempty closed convex set. We call this problem a $primal$ problem. Let f^* denote its optimal value.

Consider the following Lagrange dual problem of (1):

$$\max_{\boldsymbol{\lambda}\succeq \mathbf{0}} \min_{\mathbf{x}\in X} \mathcal{L}(\mathbf{x},\boldsymbol{\lambda}), \tag{2}$$

where $\mathcal{L} : X \times \mathbb{R}^m_+ \to \mathbb{R}$ is the Lagrangian function given by

$$\mathcal{L}(\mathbf{x},\boldsymbol{\lambda}) = f_0(\mathbf{x}) + \langle \boldsymbol{\lambda}, f(\mathbf{x})\rangle. \tag{3}$$

Let q^* denote the optimal value of (2). It is well known that the weak duality $q^* \leq f^*$ always holds, and under the Slater condition the strong duality $q^* = f^*$ also holds [12], [13].

One of the classical methods for solving problem (2) is the saddle point algorithm of Arrow-Hurwicz-Uzawa [5], also known as the Primal-Dual method. Given an initial point

This work is supported by NSF under grant ECCS #1543872, and by ONR under grant #N000141410479.

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$(\mathbf{x}_0, \boldsymbol{\lambda}_0) \in X \times \mathbb{R}_+^m$, the method iteratively updates the primal-dual variables as follows: For $t = 0, 1, \ldots$

$$\mathbf{x}_{t+1} = P_X[\mathbf{x}_t - \alpha_t \nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}_t, \boldsymbol{\lambda}_t)] \tag{4a}$$

$$\boldsymbol{\lambda}_{t+1} = [\boldsymbol{\lambda}_t + \alpha_t \nabla_{\boldsymbol{\lambda}} \mathcal{L}(\mathbf{x}_t, \boldsymbol{\lambda}_t)]^+, \tag{4b}$$

where $[\cdot]^+$ denote the coordinate-wise projection of a onto $\mathbb{R}_+^n$, $\{\alpha_t\}$ is a sequence of positive step sizes, $\nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}_t, \boldsymbol{\lambda}_t)$ and $\nabla_{\boldsymbol{\lambda}} \mathcal{L}(\mathbf{x}_t, \boldsymbol{\lambda}_t)$ are the gradients of the Lagrangian $\mathcal{L}$ at $(\mathbf{x}_t, \boldsymbol{\lambda}_t)$ with respect to $\mathbf{x}$ and $\boldsymbol{\lambda}$, respectively.

It has been extensively studied that the sequence $(\mathbf{x}_t, \boldsymbol{\lambda}_t)$ generated by the method (4a)-(4b) converges to a saddle point of the Lagrangian function $\mathcal{L}$ under some specific assumptions [5]. This and the saddle point theorem [12, Proposition 5.1.6] say that the Primal-Dual method in (4a)-(4b) solves the original problem (1).

III. Problem Statement

Consider a networked system consisting of N agents that communicate over a weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = [N]$ indexes the set of agents, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represents the pairs of communicating agents, and $\mathcal{A} \in \mathbb{R}_+^{N \times N}$ is the weighted adjacency matrix with the property that $[\mathcal{A}]_{ij} > 0$ if and only if $(i, j) \in \mathcal{E}$. We denote the set of neighbors of agent i by $\mathcal{N}_i = \{j \mid (i, j) \in \mathcal{E}\}$. Then, the associated graph Laplacian $\mathsf{L} \in \mathbb{R}^{N \times N}$ of $\mathcal{G}$ is $\mathsf{L} := \operatorname{diag}(\mathcal{A}\mathbf{1}_N) - \mathcal{A}$.

The following assumption is necessary for a proper dissemination of network-wide information.

Assumption 1: The graph $\mathcal{G}$ is connected.

By construction, L is symmetric positive semidefinite and its null space coincides with the "agreement" subspace, i.e., $\mathsf{L}\mathbf{1} = \mathbf{0}$ and $\mathbf{1}^\top \mathsf{L}^\top = \mathbf{0}^\top$. Furthermore, Assumption 1 implies that the smallest eigenvalue of L is unique and equal to zero, and for all $\mathbf{z} \in \operatorname{span}\{\mathbf{1}\}^\perp$ there exists $\lambda_2 > 0$ such that

$$\mathbf{z}^\top \mathsf{L} \mathbf{z} \geq \lambda_2 \|\mathbf{z}\|^2. \tag{5}$$

Note that this assumption ensures that λ_2, also known as the Fiedler eigenvalue of L, is always strictly positive [14].

We consider a continuous-time environment in which each agent $i \in \mathcal{V}$ selects an action trajectory $\mathbf{x}_i(t)$, $t \geq 0$ from a closed and bounded set $X \subseteq \mathbb{R}^d$. At each time t, each agent i receives a time-varying cost function $f_0^i(t, \cdot)$ chosen by nature which comes from a fixed class of convex functions $f : \mathbb{R}_+ \times X \to \mathbb{R}$, and incurs a local cost of $f_0^i(t, \mathbf{x}_i(t))$. Note that the *network-wide* cost that agents want to minimize at time t is

$$f_0(t, \mathbf{x}) = \sum_{i=1}^{N} f_0^i(t, \mathbf{x}). \tag{6}$$

We emphasize here that f_0 is neither collectively known to any of the agents nor available at any single location. In addition to this, there exist some global functions that are continuously time-varying

$$h(t, \mathbf{x}) \preceq \mathbf{0}, \tag{7}$$

where the vector function $h : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^m$ describes the constraints to satisfy at time t. Let us adopt an additional mapping $g : \mathbb{R}^m \to \mathbb{R}^m$ to define

$$f(t, \mathbf{x}) := g(h(t, \mathbf{x})).$$

Note that if $g(\mathbf{y}) = \mathbf{y}$, then $f(t, \mathbf{x})$ just coincides with $h(t, \mathbf{x})$. Otherwise, if $g(\mathbf{y}) = \max\{\mathbf{0}, \mathbf{y}\}$, $f(t, \mathbf{x})$ measures the penalty violation.

Since $f_0^i(t, \cdot)$ is the only available information for agent i at time t, each agent must determine its action $\mathbf{x}_i(t)$ based on what it "thinks" the network-wide action should be. We will postpone the details about the strategy based on which each agent i chooses such $\mathbf{x}_i(t)$ to Section IV, except to note that $\mathbf{x}_i(t)$ depends only on the local cost functions and constraints revealed up to that time, i.e., $f_0^i(s, \cdot)$ and $f(s, \cdot)$ for $s \in [0, t)$, and the coordination of the information received from the neighbors.

Consider the following *uncoordinated regret* at arbitrary $T \geq 0$:

$$\mathcal{R}(T) = \int_{t=1}^{T} \sum_{i=1}^{N} [f_0^i(t, \mathbf{x}_i(t)) - f_0^i(t, \mathbf{x}^*)] dt, \tag{8}$$

which computes the accumulated difference between the summation of the local cost incurred at each agent's own state $\mathbf{x}_i(t)$ and the best fixed point chosen from an offline and centralized view. Here $\mathbf{x}^*$ is defined as

$$\mathbf{x}^* := \inf_{\mathbf{x} \in X, h(t,\mathbf{x}) \preceq \mathbf{0}} \int_{t=1}^{T} \sum_{i=1}^{N} f_0^i(\mathbf{x}) dt, \tag{9}$$

i.e., $\mathbf{x}^*$ is a single state that could have been chosen with perfect advance knowledge of all cost functions $\{f_0^i(t, \cdot)\}_{i \in \mathcal{V}, t \geq 0}$ and all constraint functions $\{h(t, \cdot)\}_{t \geq 0}$, and without any restriction on the communication between any agents. A quantity of more interest is the following *coordinated regret* as seen by agent $j \in \mathcal{V}$ at arbitrary $T \geq 0$:

$$\mathcal{R}^j(T) = \int_{t=1}^{T} \sum_{i=1}^{N} [f_0^i(t, \mathbf{x}_j(t)) - f_0^i(t, \mathbf{x}^*)] dt. \tag{10}$$

This *coordinated regret* computes the difference between the network cost incurred by a single agent's sequence of states $\{\mathbf{x}_j(t)\}_{t=0}^T$ and the best fixed point $\mathbf{x}^*$ defined in (9). Designing a coordination algorithm which reduces $\mathcal{R}^j$ for all $j \in \mathcal{V}$ is more meaningful as this would guarantee the average cost computed at any location of the network is nearly as low as the cost of the best choice $\mathbf{x}^*$.

Since this is an online constrained problem, we also need a notion of regret associated with the constraints which is analogous to the regret in (10) associated with the cost function. We consider the following *cumulative feasibility violation* for any arbitrary $T \geq 0$ and $j \in \mathcal{V}$:

$$\mathcal{F}^j(T) = \int_{t=0}^{T} f(t, \mathbf{x}_j(t)) dt, \tag{11}$$

which can be used to show that the constraints in (7) are satisfied in the long run at any location in the network.

Our goal is to design a decentralized controller for updating each trajectory $\mathbf{x}_i(t)$ so that the generated trajectory incurs sublinear regret and constraint violation, i.e such that the coordinated regret (10) and constraint violation (11) (a) are sublinear over the time horizon T for any $j \in \mathcal{V}$ (such that the average $\mathcal{R}^j(T)/T$ and $\mathcal{F}^j(T)/T$ converge to zero), and (b) show reasonable dependence on the total number of network agents and on the topology of the communication graphs. For this paper, however, we leave the proof for the constraint violation as our future work.

IV. Distributed Online Saddle-Point Algorithm

We now introduce our proposed distributed online saddle-point controller which combines the saddle point method of Arrow-Hurwicz-Uzawa [5] with the network Laplacian.

For each $i \in \mathcal{V}$, let the penalty function $\mathcal{H}^i : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ be defined as

$$\mathcal{H}^i(t, \mathbf{x}, \boldsymbol{\lambda}) := f_0^i(t, \mathbf{x}) + \boldsymbol{\lambda}^\top f(t, \mathbf{x}), \tag{12}$$

Then, we propose local controllers for all $i \in \mathcal{V}$ that take the form of the following projected dynamical system:

$$\dot{\mathbf{x}}_i(t) = \Pi_X \left[\mathbf{x}_i(t), \sum_{j=1}^N -\mathsf{L}_{ij}\mathbf{x}_j(t) - \epsilon \mathcal{H}_x^i(t, \mathbf{x}_i(t), \boldsymbol{\lambda}_i(t)) \right], \tag{13a}$$

$$\dot{\boldsymbol{\lambda}}_i(t) = \Pi_\Lambda \left[\boldsymbol{\lambda}_i(t), \sum_{j=1}^N -\mathsf{L}_{ij}\boldsymbol{\lambda}_j(t) + \epsilon \mathcal{H}_\lambda^i(t, \mathbf{x}_i(t), \boldsymbol{\lambda}_i(t)) \right], \tag{13b}$$

where $X \subseteq \mathbb{R}^d$, $\Lambda \subseteq \mathbb{R}_+^m$ and $\epsilon > 0$ is a step size which will be later determined in the convergence proof. The vectors $\mathcal{H}_x^i(t, \mathbf{x}, \boldsymbol{\lambda}) \in \mathbb{R}^d$ and $\mathcal{H}_\lambda^i(t, \mathbf{x}, \boldsymbol{\lambda}) \in \mathbb{R}^m$ are the gradients of the local penalty function (12) with respect to $\mathbf{x}$ and $\boldsymbol{\lambda}$ which are given as

$$\mathcal{H}_x^i(t, \mathbf{x}, \boldsymbol{\lambda}) = f_{0,x}^i(t, \mathbf{x}) + f_x(t, \mathbf{x})^\top \boldsymbol{\lambda} \tag{14a}$$

$$\mathcal{H}_\lambda^i(t, \mathbf{x}, \boldsymbol{\lambda}) = f(t, \mathbf{x}). \tag{14b}$$

Furthermore, the projection operator $\Pi_X[\cdot, \cdot]$ is defined as follows.

Definition 1: (Projection of a vector at a point) Let $X \subseteq \mathbb{R}^d$ be a closed convex set. Then, for any $\mathbf{x} \in X$ and $\mathbf{v} \in \mathbb{R}^d$, we define the projection of $\mathbf{v}$ over the set X at the point $\mathbf{x}$ as

$$\Pi_X[\mathbf{x}, \mathbf{v}] = \lim_{\delta \to 0^+} \frac{P_X(\mathbf{x} + \delta \mathbf{v}) - \mathbf{x}}{\delta},$$

where $P_X(\mathbf{z}) = \arg\min_{\mathbf{y} \in X} \|\mathbf{y} - \mathbf{z}\|^2$.

As is demonstrated in [11], [15], $\Pi_X[\mathbf{x}, \mathbf{v}]$ is equivalent to $P_{\text{co}(X)}[\mathbf{v}]$ where $\text{co}(X)$ is the conical hull of X whose vertex is at the point $\mathbf{x}$. In other words, if the point $\mathbf{x}$ is in the interior of X, $\Pi_X[\mathbf{x}, \mathbf{v}]$ just becomes $\mathbf{v}$, otherwise, it becomes the component of $\mathbf{v}$ that is tangential to the set X at the point $\mathbf{x}$.

The trajectories $\mathbf{x}_i(t)$ and $\boldsymbol{\lambda}_i(t)$ are local estimates of the network-wide primal and dual points as seen by agent i. Each agent i moves its trajectories $\mathbf{x}_i(t)$ and $\boldsymbol{\lambda}_i(t)$ based on the proportional feedback on the disagreements, $\sum_{j=1}^N -\mathsf{L}_{ij}\mathbf{x}_j(t)$ and $\sum_{j=1}^N -\mathsf{L}_{ij}\boldsymbol{\lambda}_j(t)$, and the locally estimated Lagrangian $\mathcal{H}^i(t, \mathbf{x}_i(t), \boldsymbol{\lambda}_i(t))$. Therefore, the dynamics (13a)-(13b) can be seen as a continuous-time approximation of the centralized update in (4a)-(4b). Later we will see that the errors due to this estimation will behave nicely as long as $\mathbf{x}_i(t)$ and $\boldsymbol{\lambda}_i(t)$ stay close to the mean field $\bar{\mathbf{x}}(t) = \frac{1}{N}\sum_{i=1}^N \mathbf{x}_i(t)$ and $\bar{\boldsymbol{\lambda}}(t) = \frac{1}{N}\sum_{i=1}^N \boldsymbol{\lambda}_i(t)$.

Note that the dynamics in (13a)-(13b) can be rewritten compactly in a stacked vector form. For the primal dynamics (13a), let us define for all $i \in \mathcal{V}$,

$$\mathbf{d}_i(t) := \mathcal{H}_x^i(t, \mathbf{x}_i(t), \boldsymbol{\lambda}_i(t)), \tag{15a}$$

$$\begin{aligned}\mathbf{p}_i(t) := \Pi_X &\left[\mathbf{x}_i(t), \sum_{j=1}^N -\mathsf{L}_{ij}\mathbf{x}_j(t) - \epsilon \mathcal{H}_x^i(t, \mathbf{x}_i(t), \boldsymbol{\lambda}_i(t)) \right] \\ &- \sum_{j=1}^N -\mathsf{L}_{ij}\mathbf{x}_j(t) + \epsilon \mathcal{H}_x^i(t, \mathbf{x}_i(t), \boldsymbol{\lambda}_i(t)).\end{aligned} \tag{15b}$$

By stacking all the components over i, we further define $\mathbf{d}(t) := (\mathbf{d}_1(t), \ldots, \mathbf{d}_N(t))$, $\mathbf{p}(t) := (\mathbf{p}_1(t), \ldots, \mathbf{p}_N(t))$ and $\mathbf{x}(t) := (\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t))$. Then, the dynamics in (13a) can be written in a stacked vector form as

$$\dot{\mathbf{x}}(t) = -\mathbf{L}_d \mathbf{x}(t) - \epsilon \mathbf{d}(t) + \mathbf{p}(t), \tag{16}$$

where $\mathbf{L}_d := \mathsf{L} \otimes I_d$. Similarly, the dual dynamics in (13b) can be rewritten as

$$\dot{\boldsymbol{\lambda}}(t) = -\mathbf{L}_m \boldsymbol{\lambda}(t) + \epsilon \boldsymbol{\delta}(t) + \boldsymbol{\pi}(t), \tag{17}$$

where in this case $\mathbf{L}_m := \mathsf{L} \otimes I_m$, $\boldsymbol{\delta}(t) := (\boldsymbol{\delta}_1(t), \ldots, \boldsymbol{\delta}_N(t))$, $\boldsymbol{\pi}(t) := (\boldsymbol{\pi}_1(t), \ldots, \boldsymbol{\pi}_N(t))$, $\boldsymbol{\lambda}(t) := (\boldsymbol{\lambda}_1(t), \ldots, \boldsymbol{\lambda}_N(t))$ and

$$\boldsymbol{\delta}_i(t) := \mathcal{H}_\lambda^i(t, \mathbf{x}_i(t), \boldsymbol{\lambda}_i(t)), \tag{18a}$$

$$\begin{aligned}\boldsymbol{\pi}_i(t) := \Pi_X &\left[\mathbf{x}_i(t), \sum_{j=1}^N -\mathsf{L}_{ij}\boldsymbol{\lambda}_j(t) + \epsilon \mathcal{H}_\lambda^i(t, \mathbf{x}_i(t), \boldsymbol{\lambda}_i(t)) \right] \\ &- \sum_{j=1}^N -\mathsf{L}_{ij}\boldsymbol{\lambda}_j(t) - \epsilon \mathcal{H}_\lambda^i(t, \mathbf{x}_i(t), \boldsymbol{\lambda}_i(t)).\end{aligned} \tag{18b}$$

For our regret analysis, we frequently make use of the following lemma which provides a relation between the original vector $\mathbf{v}$ and its projection.

Lemma 1: [11, Lemma 1] Let $X \subseteq \mathbb{R}^d$ be a convex set and $\mathbf{x}_0, \mathbf{x} \in X$. Then, for any $\mathbf{v} \in \mathbb{R}^d$

$$(\mathbf{x}_0 - \mathbf{x})^\top \Pi_X(\mathbf{x}_0, \mathbf{v}) \le (\mathbf{x}_0 - \mathbf{x})^\top \mathbf{v}.$$

V. Regret Analysis

We now provide a regret analysis of the proposed distributed online saddle-point method. In Section V-A, we present additional assumptions on the objective and constraint functions. In Sections V-B and V-C, we present two key lemmas that will be used in the proof of the main results. The main results are provided in Section V-D.

A. Assumptions

The following assumption is typical in any gradient-based algorithms. It ensures the boundedness of the gradients so that the method in (13a)-(13b) is well behaved.

Assumption 2: (a) The sets $X \subseteq \mathbb{R}^d$ and $\Lambda \subseteq \mathbb{R}^m_+$ are convex, closed and bounded.

(b) The functions $f_0^i(t,\cdot): X \to \mathbb{R}$ for all $i \in \mathcal{V}$ and $t \geq 0$, and $f_j(t,\cdot): X \to \mathbb{R}^m$ for all $j \in [m]$ and $t \geq 0$ are convex, but not necessarily differentiable. Furthermore, they are Lipschitz continuous over X, i.e., there exist constants $L_0, L_f > 0$ such that for any $\mathbf{x}, \mathbf{y} \in X$

$$|f_0^i(t,\mathbf{x}) - f_0^i(t,\mathbf{y})| \leq L_0\|\mathbf{x}-\mathbf{y}\|,\ \forall i \in \mathcal{V},\ t \geq 0,$$
$$|f_j(t,\mathbf{x}) - f_j(t,\mathbf{y})| \leq L_f\|\mathbf{x}-\mathbf{y}\|,\ \forall j \in [m],\ t \geq 0.$$

Some direct consequences of Assumption 2 are uniformly bounded (sub)gradients, i.e., for any $\mathbf{x} \in X$

$$\|f_{0,x}^i(t,\mathbf{x})\| \leq L_0,\ \forall i \in \mathcal{V},\ t \geq 0,$$
$$\|f_{j,x}(t,\mathbf{x})\| \leq L_f,\ \forall j \in [m],\ t \geq 0,$$

and bounded function values, i.e., for any $\mathbf{x} \in X$, there exist constants $C_0, C_f > 0$ such that

$$|f_0^i(t,\mathbf{x})| \leq C_0,\ \forall i \in \mathcal{V},\ t \geq 0,$$
$$|f_j(t,\mathbf{x})| \leq C_f,\ \forall j \in [m],\ t \geq 0.$$

Furthermore, due to the boundedness of the set X and Λ, there exist some constants $C_{\mathbf{x}}, C_{\boldsymbol{\lambda}}$ such that

$$\max_{\mathbf{x}\in X}\|\mathbf{x}\| \leq C_{\mathbf{x}} \text{ and } \max_{\boldsymbol{\lambda}\in\Lambda}\|\boldsymbol{\lambda}\| \leq C_{\boldsymbol{\lambda}}.$$

Assumption 2 ensures the boundedness of the gradients defined in (14a)-(14b). Indeed, we have for any $i \in \mathcal{V}$, $t \geq 0$ and $(\mathbf{x}, \boldsymbol{\lambda}) \in X \times \Lambda$

$$\|\mathcal{H}_x^i(t,\mathbf{x},\boldsymbol{\lambda})\| = \|f_{0,x}^i(t,\mathbf{x}) + f_x(t,\mathbf{x})^\top\boldsymbol{\lambda}\| \leq L_0 + \sqrt{md}L_fC_{\boldsymbol{\lambda}} := L_{\mathbf{x}}, \quad (19a)$$
$$\|\mathcal{H}_\lambda^i(t,\mathbf{x},\boldsymbol{\lambda})\| = \|f(t,\mathbf{x})\| \leq \sqrt{m}C_f := L_{\boldsymbol{\lambda}}, \quad (19b)$$

where we defined $L_{\mathbf{x}}$ and $L_{\boldsymbol{\lambda}}$ for future reference.

B. Basic Iterate Relation

In this section, we establish some basic relation in Lemma 2 that holds for the trajectories $\mathbf{x}_i(t)$ and $\boldsymbol{\lambda}_i(t)$ obtained by the distributed online saddle-point method in (13a)-(13b). The relation will play an important role in the sequel, since it provides guidelines for choosing the step size ϵ for obtaining a sublinearly bounded regret. The proof of this lemma is omitted due to the space limitation.

For the analysis, let us define the agreement subspace $\mathcal{C}_{\mathbf{x}} := \{\mathbf{1}_N \otimes \mathbf{x} \mid \mathbf{x} \in \mathbb{R}^d\}$ and $\mathcal{C}_{\boldsymbol{\lambda}} := \{\mathbf{1}_N \otimes \boldsymbol{\lambda} \mid \boldsymbol{\lambda} \in \mathbb{R}^m\}$. Let $\mathbf{M}_d := \frac{1}{N}\mathbf{1}\mathbf{1}^\top \otimes I_d$ and $\mathbf{M}_m := \frac{1}{N}\mathbf{1}\mathbf{1}^\top \otimes I_m$ be the linear projection operators onto $\mathcal{C}_{\mathbf{x}}$ and $\mathcal{C}_{\boldsymbol{\lambda}}$, respectively. For convenience, we let K denote the Laplacian of the complete graph whose edge weights are all $\frac{1}{N}$, i.e., $\mathsf{K} = I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^\top$, and we often use $\mathbf{K}_d := \mathsf{K} \otimes I_d = I_{Nd} - \mathbf{M}_d$ and $\mathbf{K}_m := \mathsf{K} \otimes I_m = I_{Nm} - \mathbf{M}_m$. Note that $\mathbf{M}_d$, $\mathbf{M}_m$, $\mathbf{K}_d$ and $\mathbf{K}_m$ are symmetric and idempotent by construction, that is, for example $\mathbf{M}_d^2 = \mathbf{M}_d = \mathbf{M}_d^\top$.

Lemma 2: Let Assumption 2 hold. Consider the trajectories generated by the dynamics in (13a)-(13b). Then, for any $(\tilde{\mathbf{x}}, \tilde{\boldsymbol{\lambda}}) \in X \times \Lambda$ and $T \geq 0$, we have:

$$\int_{t=0}^{T}\sum_{i=1}^{N}\left[\mathcal{H}^i(t,\mathbf{x}_i(t),\tilde{\boldsymbol{\lambda}}) - \mathcal{H}^i(t,\tilde{\mathbf{x}},\boldsymbol{\lambda}_i(t))\right]dt \leq \frac{V_{\tilde{\mathbf{x}},\tilde{\boldsymbol{\lambda}}}(\mathbf{x}(0),\boldsymbol{\lambda}(0))}{\epsilon} + \sqrt{N}L_{\mathbf{x}}\int_0^T\|\mathbf{K}_d\mathbf{x}(t)\|dt + \sqrt{N}L_{\boldsymbol{\lambda}}\int_0^T\|\mathbf{K}_m\boldsymbol{\lambda}(t)\|dt,$$

where $\mathbf{x}(t) = (\mathbf{x}_1(t),\ldots,\mathbf{x}_N(t))$ and $\boldsymbol{\lambda}(t) = (\boldsymbol{\lambda}_1(t),\ldots,\boldsymbol{\lambda}_N(t))$ for all $t \geq 0$, and $V_{\tilde{\mathbf{x}},\tilde{\boldsymbol{\lambda}}}(\mathbf{x},\boldsymbol{\lambda}) : \mathbb{R}^{Nd}\times\mathbb{R}^{Nm} \to \mathbb{R}$ is the Lyapunov function defined as

$$V_{\tilde{\mathbf{x}},\tilde{\boldsymbol{\lambda}}}(\mathbf{x},\boldsymbol{\lambda}) = \frac{1}{2}\|\mathbf{M}_d\mathbf{x} - \mathbf{1}_N\otimes\tilde{\mathbf{x}}\|^2 + \frac{1}{2}\|\mathbf{M}_m\boldsymbol{\lambda} - \mathbf{1}_N\otimes\tilde{\boldsymbol{\lambda}}\|^2, \quad (20)$$

where $\mathbf{M}_d = \frac{1}{N}\mathbf{1}\mathbf{1}^\top \otimes I_d$ and $\mathbf{M}_m = \frac{1}{N}\mathbf{1}\mathbf{1}^\top \otimes I_m$.

With the choice of $(\tilde{\mathbf{x}}, \tilde{\boldsymbol{\lambda}}) := (\mathbf{x}^*, \mathbf{0})$, the LHS of Lemma 2 becomes the uncoordinated regret defined in (8). This indicates that the uncoordinated regret will remain sublinear, provided that the step size ϵ can be set in the order of $o(T)$, and the remaining two terms on the RHS can be sublinearly bounded.

Note that the last two terms in Lemma 2 respectively denote the network-wide disagreement among the local primal variables $\mathbf{x}_i(t)$'s and the local dual variables $\boldsymbol{\lambda}_i(t)$'s, accumulated over $t \in [0,T]$. In order to see why this is true, let us define the mean field $\bar{\mathbf{x}}(t) = \frac{1}{N}\sum_{i=1}^N\mathbf{x}_i(t)$ and $\bar{\boldsymbol{\lambda}}(t) = \frac{1}{N}\sum_{i=1}^N\boldsymbol{\lambda}_i(t)$, and the fluctuation $\hat{\mathbf{x}}_i(t) = \mathbf{x}_i(t) - \bar{\mathbf{x}}(t)$ and $\hat{\boldsymbol{\lambda}}_i(t) = \boldsymbol{\lambda}_i(t) - \bar{\boldsymbol{\lambda}}(t)$. Then, by defining $\hat{\mathbf{x}}(t) := (\hat{\mathbf{x}}_1(t),\ldots,\hat{\mathbf{x}}_N(t))$ and $\hat{\boldsymbol{\lambda}}(t) := (\hat{\boldsymbol{\lambda}}_1(t),\ldots,\hat{\boldsymbol{\lambda}}_N(t))$, we can see that

$$\mathbf{K}_d\mathbf{x}(t) = \hat{\mathbf{x}}(t) = \mathbf{x}(t) - \mathbf{1}_N\otimes\bar{\mathbf{x}}(t), \quad (21a)$$
$$\mathbf{K}_m\boldsymbol{\lambda}(t) = \hat{\boldsymbol{\lambda}}(t) = \boldsymbol{\lambda}(t) - \mathbf{1}_N\otimes\bar{\boldsymbol{\lambda}}(t). \quad (21b)$$

In other words, the two terms can be represented as

$$\|\mathbf{K}_d\mathbf{x}(t)\| = \left(\sum_{i=1}^N\|\mathbf{x}_i(t) - \bar{\mathbf{x}}(t)\|^2\right)^{\frac{1}{2}}, \quad (22a)$$
$$\|\mathbf{K}_m\boldsymbol{\lambda}(t)\| = \left(\sum_{i=1}^N\|\boldsymbol{\lambda}_i(t) - \bar{\boldsymbol{\lambda}}(t)\|^2\right)^{\frac{1}{2}}. \quad (22b)$$

C. Sublinear Network Disagreement

In the next lemma, we show that the network-wide disagreement terms appearing in Lemma 2 are sublinearly bounded over T.

Lemma 3: (Sublinear Network Disagreement) Let Assumptions 1-2 hold. Consider the trajectories generated by the dynamics in (13a)-(13b) and the step size ϵ is chosen as:

$$\epsilon = \frac{1}{\sqrt{T}}. \quad (23)$$

Then, for any $T \geq 0$, we have:

(a)

$$\int_{t=0}^{T} \|\mathbf{K}_d \mathbf{x}(t)\| dt \leq \frac{\|\mathbf{K}_d \mathbf{x}(0)\|}{\lambda_2} + \frac{\sqrt{N} L_{\mathbf{x}}}{\lambda_2} \sqrt{T},$$

(b)

$$\int_{t=0}^{T} \|\mathbf{K}_m \boldsymbol{\lambda}(t)\| dt \leq \frac{\|\mathbf{K}_m \boldsymbol{\lambda}(0)\|}{\lambda_2} + \frac{\sqrt{N} L_{\boldsymbol{\lambda}}}{\lambda_2} \sqrt{T},$$

where $\|\mathbf{K}_d \mathbf{x}(0)\|$ and $\|\mathbf{K}_d \mathbf{x}(0)\|$ are the total initialization errors in the primal and dual variables (see Eq. (22a)-(22b)).

Proof: **(a)** From (21a) and (16), we can write

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= \mathbf{K}_d \dot{\mathbf{x}}(t) \\ &= -\mathbf{K}_d \mathbf{L}_d (\hat{\mathbf{x}}(t) + \mathbf{1}_N \otimes \bar{\mathbf{x}}(t)) - \epsilon \mathbf{K}_d \mathbf{d}(t) + \mathbf{K}_d \mathbf{p}(t) \\ &= -\mathbf{L}_d \hat{\mathbf{x}}(t) - \epsilon \mathbf{K}_d \mathbf{d}(t) + \mathbf{K}_d \mathbf{p}(t), \end{aligned} \tag{24}$$

where in the last inequality we used relations $\mathbf{K}_d \mathbf{L}_d = \mathbf{L}_d$ and $\mathbf{L}_d \mathbf{1}_N \otimes \bar{\mathbf{x}}(t) = 0$. Consider a Lyapunov candidate $V(\hat{\mathbf{x}}) = \frac{1}{2} \hat{\mathbf{x}}^\top \hat{\mathbf{x}}$. Then, from relation (21a), we can write

$$V(\hat{\mathbf{x}}(t)) = \frac{1}{2} \hat{\mathbf{x}}(t)^\top \hat{\mathbf{x}}(t) = \frac{1}{2} \|\mathbf{K}_d \mathbf{x}(t)\|^2. \tag{25}$$

Using $\hat{\mathbf{x}}(t)^\top \mathbf{K}_d = \hat{\mathbf{x}}(t)^\top$ in the dynamics of $\hat{\mathbf{x}}(t)$ in (24), the derivative of $V(\hat{\mathbf{x}}(t))$ with respect to time is given by

$$\begin{aligned} \dot{V}(\hat{\mathbf{x}}(t)) &= \hat{\mathbf{x}}(t)^\top \dot{\hat{\mathbf{x}}}(t) \\ &= -\hat{\mathbf{x}}(t)^\top \mathbf{L}_d \hat{\mathbf{x}}(t) - \epsilon \hat{\mathbf{x}}(t)^\top \mathbf{d}(t) + \hat{\mathbf{x}}(t)^\top \mathbf{p}(t). \end{aligned} \tag{26}$$

We now find an upper-bound for each term on the RHS of (26). For the first term, from (5) and the fact $\hat{\mathbf{x}}(t) \in \text{span}\{\mathbf{1}_{Nd}\}^\perp$, it is straightforward that

$$\begin{aligned} \hat{\mathbf{x}}(t)^\top \mathbf{L}_d \hat{\mathbf{x}}(t) &\geq \lambda_2 \hat{\mathbf{x}}(t)^\top \hat{\mathbf{x}}(t) \\ &= \lambda_2 \|\hat{\mathbf{x}}(t)\|^2 = 2\lambda_2 V(\hat{\mathbf{x}}(t)). \end{aligned} \tag{27}$$

For the second term on the RHS of (26), we obtain the following chain of relations:

$$\hat{\mathbf{x}}(t)^\top \mathbf{d}(t) = \sum_{i=1}^{N} \hat{\mathbf{x}}_i(t)^\top \mathbf{d}_i(t) \tag{28a}$$

$$= \sum_{i=1}^{N} (\mathbf{x}_i(t) - \bar{\mathbf{x}}(t))^\top \mathcal{H}_x^i(t, \mathbf{x}_i(t), \boldsymbol{\lambda}_i(t)) \tag{28b}$$

$$\geq -L_{\mathbf{x}} \sum_{i=1}^{N} \|\mathbf{x}_i(t) - \bar{\mathbf{x}}(t)\|, \tag{28c}$$

where (28a)-(28b) follow from the definition of $\mathbf{d}(t)$ in (15a); (28c) follows from Assumption 2 and relation (19a). Using the Jensen's inequality, (28) can be further bounded as:

$$-\epsilon \hat{\mathbf{x}}(t)^\top \mathbf{d}(t) \leq \epsilon L_{\mathbf{x}} \sum_{i=1}^{N} \|\mathbf{x}_i(t) - \bar{\mathbf{x}}(t)\| \tag{29}$$

$$\leq \epsilon L_{\mathbf{x}} \sqrt{N \sum_{i=1}^{N} \|\mathbf{x}_i(t) - \bar{\mathbf{x}}(t)\|^2} = \epsilon L_{\mathbf{x}} \sqrt{2N} \sqrt{V(\hat{\mathbf{x}}(t))}.$$

For the third term on the RHS of (26), using the definition of $\mathbf{p}_i(t)$ in (15b) and Lemma 1, we obtain

$$\hat{\mathbf{x}}(t)^\top \mathbf{p}(t) = \sum_{i=1}^{N} \hat{\mathbf{x}}_i(t)^\top \mathbf{p}_i(t) \leq 0. \tag{30}$$

Combining the bounds in (27), (29) and (30) into the RHS of (26), we arrive at:

$$\dot{V}(\hat{\mathbf{x}}(t)) \leq -2\lambda_2 V(\hat{\mathbf{x}}(t)) + \epsilon L_{\mathbf{x}} \sqrt{2N} \sqrt{V(\hat{\mathbf{x}}(t))}.$$

Let $\mathbf{q}(t) := \sqrt{V(\hat{\mathbf{x}}(t))}$. Then, the above differential inequality can be rewritten as

$$\dot{\mathbf{q}}(t) \leq -\lambda_2 \mathbf{q}(t) + \frac{\epsilon L_{\mathbf{x}} \sqrt{2N}}{2}.$$

Solving this first-order differential inequality yields

$$\mathbf{q}(t) \leq e^{-\lambda_2 t} \left(\mathbf{q}(0) - \frac{\epsilon L_{\mathbf{x}} \sqrt{2N}}{2\lambda_2} \right) + \frac{\epsilon L_{\mathbf{x}} \sqrt{2N}}{2\lambda_2}.$$

Using this result and relation (25) and omitting the negative term, we can write:

$$\begin{aligned} \|\mathbf{K}_d \mathbf{x}(t)\| &= \sqrt{2} \mathbf{q}(t) \\ &\leq \sqrt{2} e^{-\lambda_2 t} \mathbf{q}(0) + \frac{\epsilon L_{\mathbf{x}} \sqrt{N}}{\lambda_2}. \end{aligned}$$

Lastly, using the definition of ϵ in (23) and integrating the above relation over time 0 to T yields the desired result. ∎

Proof: **(b)** Noticing the similarity between (21a) and (21b), and (16) and (17), we omit the proof of this part since it is almost identical to that of part (a). ∎

This lemma captures the effect of the underlying network topology via the spectral gap λ_2 (also known as the Fiedler value), which encodes the algebraic connectivity of the graph $\mathcal{G}$.

D. Main Results

By combining Lemma 2 and 3, we can now provide our main results. In the next theorem, we show that the coordinated regret (cf. definition in (10)) is sublinearly bounded.

Theorem 1: (Sublinear Coordinated Regret) Let Assumptions 1-2 hold. Consider the trajectories generated by the dynamics in (13a)-(13b), where the step size ϵ is chosen as $\epsilon = \frac{1}{\sqrt{T}}$. Then, for any $T \geq 0$ and $j \in \mathcal{V}$, we have:

$$\mathcal{R}^j(T) \leq C_1 + C_2 \sqrt{T},$$

where

$$C_1 = \frac{(\sqrt{N} L_{\mathbf{x}} + \sqrt{2} L_0 N)}{\lambda_2} \|\mathbf{K}_d \mathbf{x}(0)\| + \frac{\sqrt{N} L_{\boldsymbol{\lambda}}}{\lambda_2} \|\mathbf{K}_m \boldsymbol{\lambda}(0)\|,$$

$$C_2 = V_{\mathbf{x}^*, \mathbf{0}}(\mathbf{x}(0), \boldsymbol{\lambda}(0)) + \frac{(N L_{\mathbf{x}}^2 + \sqrt{2} L_0 L_{\mathbf{x}} N^{3/2})}{\lambda_2} + \frac{N L_{\boldsymbol{\lambda}}^2}{\lambda_2},$$

and the constants $\|\mathbf{K}_d \mathbf{x}(0)\|$, $\|\mathbf{K}_m \boldsymbol{\lambda}(0)\|$, $V_{\mathbf{x}^*, \mathbf{0}}(\mathbf{x}(0), \boldsymbol{\lambda}(0))$ are defined in (22a), (22b) and (20), respectively.

Proof: Note that we can write

$$\mathcal{R}^j(T) = \int_{t=0}^{T} \sum_{i=1}^{N} [f_0^i(t, \mathbf{x}_j(t)) - f_0^i(t, \mathbf{x}_i(t))] dt$$

$$+\int_{t=0}^{T}\sum_{i=1}^{N}[f_0^i(t,\mathbf{x}_i(t))-f_0^i(t,\mathbf{x}^*)]dt. \qquad (31)$$

The first term on the RHS of (31) can be understood as the discrepancy between the coordinated regret and the uncoordinated regret. Using Assumption 2, this term can be bounded as:

$$\sum_{i=1}^{N}[f_0^i(t,\mathbf{x}_j(t))-f_0^i(t,\mathbf{x}_i(t))]\leq L_0\sum_{i=1}^{N}\|\mathbf{x}_j(t)-\mathbf{x}_i(t)\|. \qquad (32)$$

This can be further bounded using the Jensen's inequality:

$$\left(\sum_{i=1}^{N}\|\mathbf{x}_j(t)-\mathbf{x}_i(t)\|\right)^2\leq N\sum_{i=1}^{N}\|\mathbf{x}_j(t)-\mathbf{x}_i(t)\|^2$$
$$\leq N\sum_{j=1}^{N}\sum_{i=1}^{N}\|\mathbf{x}_j(t)-\mathbf{x}_i(t)\|^2. \qquad (33)$$

Furthermore, from the fact that $\mathbf{K}_d^2=\mathbf{K}_d=\mathbf{K}_d^\top$, we have

$$N\|\mathbf{K}_d\mathbf{x}(t)\|^2=N\mathbf{x}(t)^\top\mathbf{K}_d\mathbf{x}(t)$$
$$=\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\|\mathbf{x}_i(t)-\mathbf{x}_j(t)\|^2. \qquad (34)$$

Combining the above two relations, (32) can be rewritten as

$$\sum_{i=1}^{N}[f_0^i(t,\mathbf{x}_j(t))-f_0^i(t,\mathbf{x}_i(t))]\leq\sqrt{2}L_0N\|\mathbf{K}_d\mathbf{x}(t)\|. \qquad (35)$$

The second term on the RHS of (31) can be bounded using Lemma 2. By setting $\tilde{\mathbf{x}}=\mathbf{x}^*$ (cf. the definition of $\mathbf{x}^*$ in (9)) and $\tilde{\boldsymbol{\lambda}}=\mathbf{0}$, the LHS of Lemma 2 can be lower bounded as

$$\int_{t=0}^{T}\sum_{i=1}^{N}\left[\mathcal{H}^i(t,\mathbf{x}_i(t),\mathbf{0})-\mathcal{H}^i(t,\mathbf{x}^*,\boldsymbol{\lambda}_i(t))\right]dt$$
$$\geq\int_{t=0}^{T}\sum_{i=1}^{N}\left[f_0^i(t,\mathbf{x}_i(t))-f_0^i(t,\mathbf{x}^*)\right]dt, \qquad (36)$$

where the inequality follows from the fact that $\mathbf{0}^\top f(t,\mathbf{x}_i(t))=0$ and $\boldsymbol{\lambda}_i(t)^\top f(t,\mathbf{x}^*)\leq 0$. Therefore, substituting (35) and (36) in (31), and using Lemma 2 with $(\tilde{\mathbf{x}},\tilde{\boldsymbol{\lambda}})=(\mathbf{x}^*,\mathbf{0})$, we obtain

$$\mathcal{R}^j(T)\leq\int_{t=0}^{T}\sum_{i=1}^{N}\left[\mathcal{H}^i(t,\mathbf{x}_i(t),\mathbf{0})-\mathcal{H}^i(t,\mathbf{x}^*,\boldsymbol{\lambda}_i(t))\right]dt$$
$$+\sqrt{2}L_0N\int_{t=0}^{T}\|\mathbf{K}_m\mathbf{x}(t)\|dt$$
$$\leq\frac{V_{\mathbf{x}^*,\mathbf{0}}(\mathbf{x}(0),\boldsymbol{\lambda}(0))}{\epsilon}$$
$$+(\sqrt{N}L_{\mathbf{x}}+\sqrt{2}L_0N)\int_0^T\|\mathbf{K}_d\mathbf{x}(t)\|dt$$
$$+\sqrt{N}L_{\boldsymbol{\lambda}}\int_0^T\|\mathbf{K}_m\boldsymbol{\lambda}(t)\|dt$$

Lastly, using Lemma 3 for bounding the network disagreement terms and the definition of ϵ in (23), we obtain the desired result. ■

The bound shows that, for any connected graph $\mathcal{G}$, the worst-case regret of the proposed saddle-point controller (13a)-(13b) is of order $O(\sqrt{T})$. It also captures the dependence on the properties of the underlying network (encoded in λ_2), the total initialization error, as well as the number of nodes N. Provided that λ_2 does not depend on N (such as expanders, for example), the regret bound grows in the order of $N^{3/2}$.

VI. Conclusion and Future Work

We proposed a distribute continuous-time variant of the saddle-point method for solving online convex optimization problems defined over networks. We analyzed that the algorithm achieves regret of order $O(\sqrt{T})$ under some typical assumptions like the Lipschitz continuity of the objective and constraint functions, and the network connectivity. Future work will include the analysis of the regret in constraint violation and numerical results on some practical applications. Our proposed algorithm can be extended in multiple ways, including the study of practical implementation concerning switching network topologies, asynchronism, channel noise and communication delays.

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