Graph Theoretic Connectivity Control of Mobile Robot Networks

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***Abstract*— In this paper we provide a theoretical framework for controlling graph connectivity in mobile robot networks. We discuss proximity-based communication models composed of disk-based or uniformly-fading-signal-strength communication links. A graph theoretic definition of connectivity is provided, as well as an equivalent definition based on algebraic graph theory, which employs the adjacency and Laplacian matrices of the graph and their spectral properties. Based on these results, we discuss centralized and distributed algorithms to maintain, increase, and control connectivity in mobile robot networks. The various approaches discussed in this paper range from convex optimization and subgradient descent algorithms, for the maximization of the algebraic connectivity of the network, to potential fields and hybrid systems that maintain communication links or control the network topology in a least restrictive manner. Common to these approaches is the use of mobility to control the topology of the underlying communication network. We discuss applications of connectivity control to multi-robot rendezvous, flocking and formation control, where so far, network connectivity has been considered an assumption.**

***Index Terms*— Graph connectivity, algebraic graph theory, convex and subgradient optimization, hybrid systems.**

I. INTRODUCTION

MOBILE robot networks have recently emerged as an inexpensive and robust way of addressing a wide variety of tasks ranging from exploration, surveillance and reconnaissance, to cooperative construction and manipulation. The success of these stories relies on efficient information exchange and coordination between the members of the team. In fact, recent work on distributed consensus and state agreement has strongly depended on multi-hop communication for convergence and performance guarantees [1]–[14].

Multi-hop communication in multi-robot systems has typically relied on constructs from graph theory, with weighted proximity and disc-based graphs gaining the most popularity. Besides their simplicity, these models owe their popularity to their resemblance to radio signal strength models, where the signals attenuate with the distance [15]–[17]. In this context, multi-hop communication becomes equivalent to network connectivity, defined as the property of a graph to transmit information between any pair of its nodes.

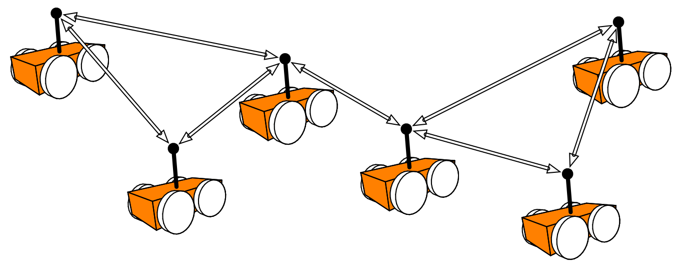

Fig. 1. Networks have long served as models of local interactions in the field mobile robotics. Robots are typically associated with the nodes of a graph and communication links with the edges.

Network connectivity has been widely studied in the area of wireless and ad-hoc networks. Of great importance in this field is the power management of the nodes for optimal routing and lifetime of the network, while ensuring connectivity [18]–[23]. This research has given rise to connectivity or topology control algorithms that regulate the transmission power of the nodes and, therefore, their communication range. Approaches range from cone-based [24]–[26] to distributed algorithms that do not involve any position information of the nodes [27], [28]. Related is also work on asymptotic bounds on the number of neighbors required to ensure connectivity in randomly deployed networks [29], as well as on the critical interference above which connectivity is lost [30]. However, this type of work focuses more on the power consumption and routing problem than the actuation and control.

Although networks have long served as models of local interactions in the field of mobile robotics (Fig. 1), until recently their structural properties have been assumed and decoupled from the control objectives, as in the case of connectivity in distributed consensus [1]–[14]. A first attempt to control the network structure was with the design of networks with maximal connectivity, where eigenvector structure-based approaches for tree networks [31], [32] were followed by optimization-based approaches applied to more general networks [33], [34]. These results were derived for static, state-independent, networks. Recently, controllability frameworks for state-dependent graphs were also proposed [35]. Nevertheless, the first work to treat connectivity as a control objective was [36], in the context of multi-robot rendezvous. Since then, a large amount of research has been targeted in this direction, and a wide range of applications and solution techniques have been proposed.

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A metric that is typically employed to capture connectivity of robotic networks is the second smallest eigenvalue $\lambda_2(\mathcal{L})$ of the Laplacian matrix $\mathcal{L}$ of the graph, also known as the algebraic connectivity or Fiedler value of the graph. It is well known that $\lambda_2(\mathcal{L})$ is a concave function of the Laplacian matrix, and when positive definite, it implies network connectivity [37]–[40]. This has given rise to optimization-based connectivity controllers, that rely on maximization of the Fiedler value [41], [42]. Since $\lambda_2(\mathcal{L})$ is a function of the network's structure via the Laplacian matrix, connectivity algorithms that relied on it were initially centralized [41]. Only recently have there been subgradient algorithms for its distributed optimization [42]. Furthermore, the Fiedler value is a non-differentiable function of the Laplacian matrix, which presents difficulties in designing feedback controllers to maintain it positive definite. Ways to overcome this problem involve either positive definiteness constraints on the determinant of the Laplacian matrix that is a differentiable function of the Laplacian [43], or distributed consensus on either Laplacian eigenvectors [44], [45] or on the network structure itself [46] for local estimation of the Fiedler value of the overall network.

Alternatively, connectivity can be captured by the sum of powers $\sum_{k=0}^{K} \mathcal{A}^k$ of the adjacency matrix $\mathcal{A}$ of the network, which represents the number of paths up to length K between every pair of nodes in the graph [40]. By definition of graph connectivity, if this number is positive definite for $K = n - 1$ and all pairs of nodes, then the network is connected (n denotes the number of nodes). For originally connected networks, maintaining positive definiteness of all positive entries of $\sum_{k=0}^{K} \mathcal{A}^k$ for any $K \leq n - 1$, maintains paths of maximum length K between agents and, as shown in [47], is sufficient to maintain connectivity of the network. This typically gives rise to optimization-based connectivity controllers [47], [48], that are often centralized due to the multihop agent dependencies that are introduced by the powers of the adjacency matrix. Since smaller powers correspond to shorter dependencies (paths), distribution is possible as K decreases. If $K = 1$, connectivity maintenance reduces to preserving the links of a connected spanning subgraph of the network and due to differentiability of the adjacency matrix, often results in feedback solution techniques. Discrete-time approaches are discussed in [36], [49], [50], while [51]–[56] rely on local gradients that may also incorporate switching in the case of link additions. Switching between arbitrary spanning topologies has also been studied with the spanning subgraphs being updated by local auctions [46], distributed spanning tree algorithms [57], combination of information dissemination algorithms and graph picking games [58], or intermediate rendezvous [59], [60]. This class of approaches are typically hybrid, combining continuous link maintenance and discrete topology control. The algebraic connectivity $\lambda_2(\mathcal{L})$ and number of paths $\sum_{k=0}^{K} \mathcal{A}^k$ metrics can also be combined to give controllers that maintain connectivity, while enforcing desired multi-hop neighborhoods for all agents [61].

The results discussed above have been successfully applied to multiple scenarios that require network connectivity to achieve a global coordinated objective. Indicative of this work is recent literature on connectivity preserving rendezvous [36], [52], [56], [62], [63], flocking [55], [64] and formation control [56], [59], where so far connectivity had been an assumption. Further extensions and contributions involve connectivity control for double integrator agents [49], agents with bounded inputs [65]–[67] and indoor navigation [61], as well as for communication based on radio signal strength [68]–[71] and visibility constraints [36], [62], [72]–[74]. Periodic connectivity for robot teams that need to occasionally split in order to achieve individual objectives [75] and sufficient conditions for connectivity in leader-follower networks [76], also add to the list. Early experimental results have demonstrated efficiency of these algorithms also in practice [75], [77], [78].

In this paper, we focus on the works of [41]–[43], [46], [56], [64], since they are the first to have formally addressed connectivity control of mobile networks for a wide range of applications and solution techniques. Our contribution is to present a cohesive overview of the key results in these papers in a unified framework. This includes basic notions of network connectivity and control theoretic methods for connectivity guarantees and convergence. The results discussed in this work incorporate a variety of mathematical tools, ranging from spectral graph theory and semidefinite programming, to gradient descent algorithms and hybrid systems. A byproduct of this work is to classify the available literature with respect to the connectivity metrics and solution techniques and provide a basis upon which future research can be built.

The rest of this paper is organized as follows. In Section II we develop graph theoretic models of communication and discuss network connectivity. In Section III we present centralized [41] and distributed [42] optimization-based approaches to maximizing the algebraic connectivity of a network, while in Section IV, we discuss gradient-based feedback controllers that rely on the spectral properties of the network [43]. In Section V we introduce distributed hybrid solutions to the problem [46], [56], while in Section VI we discuss application of connectivity control to connectivity preserving rendezvous [56], flocking [64], and formation control [56].

II. Connectivity in Mobile Robot Networks

Consider n points robots in $\mathbf{R}^d$ and let $x_i(t) \in \mathbf{R}^d$ denote the position of robot i at time $t \geq 0$. The robots can be described by either single integrator models

$$\dot{x}_i(t) = u_i(t), \tag{1}$$

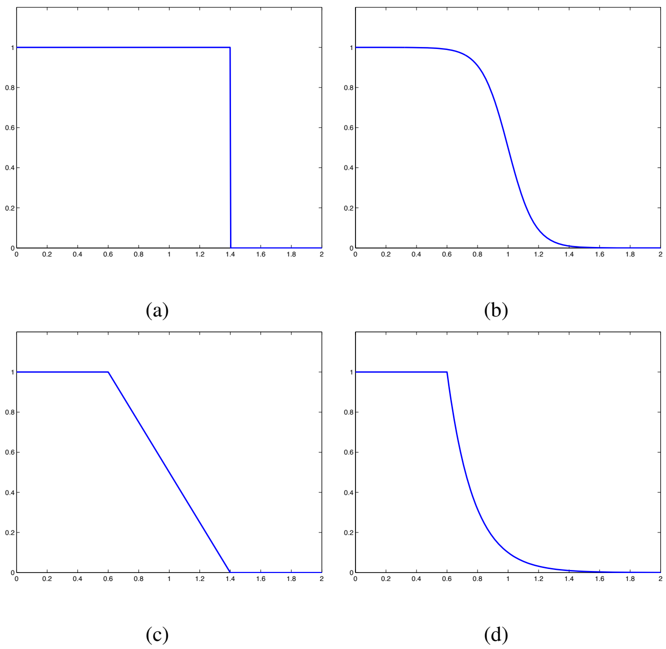
where $u_i(t) \in \mathbf{R}^d$ denotes the control input to robot i at time t, or double integrator models

$$\dot{x}_i(t) = v_i(t), \tag{2a}$$

$$\dot{v}_i(t) = u_i(t), \tag{2b}$$

where $v_i(t) \in \mathbf{R}^d$ denotes the velocity of robot i at time t. Assume further that the robots have integrated wireless communication capabilities and denote by (i, j) a communication link between robots i and j. With every communication link (i, j) we associate a weight function

$$w : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}_+,$$



(a) (b) (c) (d)

Fig. 2. Different choices for the function f. In particular, Fig. 2(a): $f(y) = 1$ if $y \leq \rho_2$, Fig. 2(b): $f(y) = \frac{1}{1+e^{-\alpha(y-\rho)}}$ with $\alpha = \frac{2}{\rho_2-\rho_1} \log\left(\frac{1-\epsilon}{\epsilon}\right)$ and $\rho = \frac{\rho_1+\rho_2}{2}$, Fig. 2(c): $f(y) = \frac{1}{\rho_1-\rho_2}y - \frac{\rho_2}{\rho_1-\rho_2}$ if $\rho_1 \leq y < \rho_2$, and Fig. 2(d): $f(y) = e^{-\alpha(y-\rho_1)}$ if $y > \rho_1$, with $\alpha = \frac{1}{\rho_2-\rho_1}\log\left(\frac{1}{\epsilon}\right)$. The above plots are for $\rho_1 = .6$, $\rho_2 = 1.4$ and $\epsilon = .01$.

such that

$$w_{ij}(t) = w(x_i(t), x_j(t)) = f(\|x_{ij}(t)\|_2), \tag{3}$$

for some $f : \mathbf{R}_+ \to \mathbf{R}_+$, where $x_{ij}(t) = x_i(t) - x_j(t)$.[1] We choose the function f to be a decreasing function of the inter-robot distance $\|x_{ij}(t)\|_2$ such that

$$1 - \epsilon < f(\|x_{ij}(t)\|_2) \leq 1, \quad \text{if} \quad \|x_{ij}(t)\|_2 < \rho_1$$

and

$$0 \leq f(\|x_{ij}(t)\|_2) < \epsilon, \quad \text{if} \quad \|x_{ij}(t)\|_2 > \rho_2,$$

for $0 < \rho_1 < \rho_2$ and small enough $0 < \epsilon < 1$ (Fig. 2). This definition captures the fact that signal strength between wireless robots is strong up to a distance ρ_1 and then decreases rapidly until it practically vanishes beyond a distance ρ_2.

The system described above gives rise to a weighted state-dependent graph

$$\mathbb{G} = (\mathbb{V}, \mathbb{W}),$$

where $\mathbb{V} = \{1, \ldots, n\}$ denotes the set of nodes indexed by the set of robots and $\mathbb{W} : \mathbb{V} \times \mathbb{V} \times \mathbf{R}_+ \to \mathbf{R}_+$ denotes the set of edge weights, such that

$$\mathbb{W}(i, j, t) = w_{ij}(t)$$

for $i, j \in \mathbb{V}$ and with $w_{ij}(t)$ as in (3). The set $\vec{\mathbb{E}}(t) = \{(i,j) \mid w_{ij}(t) > 0\}$ is called the set of directed edges of $\mathbb{G}$, while the unordered pair $\{i, j\}$ is an edge of $\mathbb{G}$ if $w_{ij}(t) > 0$ or $w_{ji}(t) > 0$. If $w_{ij}(t) = 0$ implies $w_{ji}(t) = 0$ for all $i, j \in \mathbb{V}$, then the weights are called weakly symmetric and the graph is called undirected. On the other hand, if $w_{ij}(t) = w_{ji}(t)$ for all $i, j \in \mathbb{V}$, then the weights are called symmetric. Clearly, if a graph has symmetric weights, then it is also undirected. Throughout this paper we assume graphs $\mathbb{G}$ with symmetric weights that additionally have no loops, i.e., $w_{ii}(t) = 0$ for all $i \in \mathbb{V}$. We also define the set of neighbors of node $i \in \mathbb{V}$ by $\mathbb{N}_i(t) = \{j \in \mathbb{V} \mid (i,j) \in \vec{\mathbb{E}}(t)\}$, which in the case of undirected graphs results in a mutual adjacency relationship between nodes, i.e., if $i \in \mathbb{N}_j(t)$ then $j \in \mathbb{N}_i(t)$. Similarly, we define a directed path of length k by a sequence of $k+1$ distinct nodes $i_0, i_1, \ldots, i_k \in \mathbb{V}$ such that $(i_{p-1}, i_p) \in \vec{\mathbb{E}}(t)$ for all $1 \leq p \leq k$. If the graph $\mathbb{G}$ is undirected, then so are its paths. An important topological invariant of graphs is graph connectivity, which for the case of undirected graphs is defined as follows:

Definition 2.1 (Graph connectivity): We say that an undirected graph $\mathbb{G}$ is connected if for every pair of nodes there exists a path starting at one node and ending at the other.

Network connectivity is an important property of robotic networks designed to achieve global coordinated objectives, since it ensures information sharing via multi-hop communication paths between members of the team. This property can be efficiently captured using an equivalent algebraic representation of graphs by the adjacency and Laplacian matrices.

A. Algebraic Definitions of Connectivity

We define the adjacency matrix $\mathcal{A}(t) \in \mathbf{R}_+^{n \times n}$ of the weighted graph $\mathbb{G}$ with entries

$$[\mathcal{A}(t)]_{ij} = w_{ij}(t). \tag{4}$$

Clearly, if the network has symmetric weights, then the adjacency matrix is a symmetric matrix. Furthermore, if the weights satisfy $w_{ij}(t) \in \{0, 1\}$ (Fig. 2(a)), then the powers of the adjacency matrix of a graph are closely related to network connectivity. In particular, we have the following result [40]:

Theorem 2.2 (Graph connectivity): The entry $\left[\mathcal{A}^k(t)\right]_{ij}$ of the matrix $\mathcal{A}^k(t)$ is the number of paths of length k from node i to node j in $\mathbb{G}$. Therefore, the graph $\mathbb{G}$ is connected if and only if there exists an integer K such that all the entries of the matrix $\mathcal{C}_K(t) = \sum_{k=0}^{K} \mathcal{A}^k(t)$ are non-zero.

Note that the integer K in Theorem 2.2 is upper bounded by $n-1$, since this is the length of the longest possible path in a network of n nodes. Note also that for any $K \leq n-1$ the inequality

$$[\mathcal{C}_K(t)]_{ij} > 0$$

enforces paths of maximum length K between nodes i and j in $\mathbb{V}$. It is shown in [47] that, for initially connected networks, requiring that $[C_K(t)]_{ij} > 0$, for any $K \leq n-1$, whenever $[C_K(0)]_{ij} > 0$ is sufficient for network connectivity for all time $t \geq 0$. This result can be easily understood if applied for $K = 1$, where it states that maintaining all 1-hop links of an originally connected network is sufficient for connectivity for all time. In what follows, when relying on the matrix $\mathcal{C}_K(t)$ to ensure connectivity, we only consider the case $K = 1$. The general case is discussed in [47], [48].

[1] We denote by $\mathbf{R}_+$ the set $[0, \infty)$ and by $\mathbf{R}_{++}$ the set $(0, \infty)$.

Alternatively, graph connectivity can be captured using the Laplacian matrix $\mathcal{L}(t) \in \mathbf{R}^{n\times n}$ of the network $\mathbb{G}$, which is defined by

$$[\mathcal{L}(t)]_{ij} = \begin{cases} -w_{ij}(t), & \text{if } i \neq j \\ \sum_{s\neq i} w_{is}(t), & \text{if } i = j \end{cases}. \tag{5}$$

If $\mathcal{D}(t) = \text{diag}\left(\sum_{j=1}^{n} w_{ij}(t)\right)$ denotes the diagonal matrix of degrees of the network, also called the Valency matrix of $\mathbb{G}$, then the Laplacian matrix can be written as

$$\mathcal{L}(t) = \mathcal{D}(t) - \mathcal{A}(t).$$

The Laplacian matrix of a network $\mathbb{G}$ with symmetric weights is always a symmetric positive semidefinite matrix with spectral properties closely related to network connectivity, as it can be seen from the following theorem [40]:

Theorem 2.3: Let

$$0 \leq \lambda_1(\mathcal{L}(t)) \leq \lambda_2(\mathcal{L}(t)) \leq \cdots \leq \lambda_n(\mathcal{L}(t))$$

be the ordered eigenvalues of the Laplacian matrix $\mathcal{L}(t)$. Then, $\lambda_1(\mathcal{L}(t)) = 0$ with corresponding eigenvector $\mathbf{1}$, i.e., the $n \times 1$ vector of all entries equal to 1. Moreover, $\lambda_2(\mathcal{L}(t)) > 0$ if and only if $\mathbb{G}$ is connected.

Besides an indicator of connectivity, the second smallest eigenvalue $\lambda_2(\mathcal{L}(t))$ of the Laplacian matrix of $\mathbb{G}$, also called the algebraic connectivity or Fiedler value of the network, is also a measure of the robustness of the network to link failures, captured by the notion of k-connectivity [40]:

Definition 2.4 (k-connectivity): Let $\eta(\mathbb{G})$ be the minimum number of edges that if removed from $\mathbb{G}$ increase its number of connected components. Then, for any $k \leq \eta(\mathbb{G})$ the undirected graph $\mathbb{G}$ is called k-connected.

The edge connectivity $\eta(\mathbb{G})$ and algebraic connectivity $\lambda_2(\mathcal{L}(t))$ are related by the inequality [40]

$$\lambda_2(\mathcal{L}(t)) \leq \eta(\mathbb{G}).$$

Therefore, if $\lambda_2(\mathcal{L}(t)) > k - 1$, then the network $\mathbb{G}$ is k-connected. Note that if $k = 1$, then k-connectivity reduces to the usual definition of connectivity (Definition 2.1). The results discussed above give rise to the following statement of the connectivity control problem:

Problem 1 (Network connectivity control): Given an initially connected state-dependent network $\mathbb{G}$, design distributed controllers $\{u_i(t)\}_{i=1}^n$ for the robots so that the closed loop system (1) or (2) guarantees that $\mathbb{G}$ is k-connected for all time.

In what follows, we discuss optimization [41], [42] and feedback-based [43], [46], [56] solutions to Problem 1 that employ both connectivity metrics developed above, i.e., the adjacency matrix $\mathcal{A}(t)$ and its powers as well as the algebraic connectivity $\lambda_2(\mathcal{L}(t))$. We unify these approaches under a common control framework and characterize them with respect to the amount of distribution they possess.

III. Optimization-Based Connectivity Control

Observe that $\lambda_2(\mathcal{L}(t))$ is a concave function of $\mathcal{L}(t)$ in the space $\mathbf{1}^\perp$ given by the infimum of a set of linear functions in $\mathcal{L}(t)$, i.e.,

$$\lambda_2(\mathcal{L}(t))z^T z \leq z^T \mathcal{L}(t) z$$

for all $z \in \mathbf{1}^\perp$, or equivalently

$$\lambda_2(\mathcal{L}(t)) = \inf_{z\in\mathbf{1}^\perp} \frac{z^T\mathcal{L}(t)z}{z^T z}. \tag{6}$$

Therefore, maximization of $\lambda_2(\mathcal{L}(t))$ gives rise to optimization-based approaches to the connectivity control problem. In other words, a sufficient solution to Problem 1 can be obtained by solving the optimization problem

$$\max_{x\in\mathbf{R}^{dn}} \lambda_2(\mathcal{L}(x)), \tag{7}$$

where $x = [x_1 \ x_2 \ \dots \ x_n]^T \in \mathbf{R}^{dn}$ denotes the vector of all robot positions. The two approaches to this problem that we discuss rely on concavity of the state-independent problem

$$\max_{\mathcal{L}\in\mathbf{S}^n} \lambda_2(\mathcal{L}), \tag{8}$$

to obtain an equivalent convex formulation, and then propose centralized and distributed iterative algorithms, respectively, to introduce the nonconvex dependence on the state $x \in \mathbf{R}^{dn}$.

A. Centralized Connectivity Maximization

The key idea behind a centralized solution to problem (8) is to employ the following result that relates positive definiteness of the algebraic connectivity to positive definiteness of a quadratic expression of the Laplacian matrix [41].

Proposition 3.1: Let $\mathcal{P} = [p_1 \ \dots \ p_{n-1}] \in \mathbf{R}^{n\times n-1}$, be such that $p_i^T\mathbf{1} = 0$ for all $i = 1, \dots, n-1$ and $p_i^T p_j = 0$ for all $i \neq j$. Then, $\lambda_2(\mathcal{L}) > 0$ if and only if $\mathcal{P}^T\mathcal{L}\mathcal{P} \succ 0$.

Proof: Since, for any graph we have that $\mathcal{L} \succeq 0$ and $\mathcal{L}\mathbf{1} = \mathbf{0}$, the smallest eigenvalue $\lambda_1(\mathcal{L}) = 0$ is always zero and $\text{rank}(\mathcal{L}) \leq n-1$. This implies that $\lambda_2(\mathcal{L}) > 0$ if and only if $w^T\mathcal{L}w > 0$ for all $w \in \mathbf{1}^\perp$.

Let $z \in \mathbf{R}^{n-1}$ and consider the quadratic form $z^T\mathcal{P}^T\mathcal{L}\mathcal{P}z = (\mathcal{P}z)^T\mathcal{L}\mathcal{P}z$. Let $w = \mathcal{P}z$. Since $\mathcal{P}$ is full rank and $\mathbf{1}^T w = \mathbf{1}^T\mathcal{P}z = 0$ for any $z \in \mathbf{R}^{n-1}$, $w = \mathcal{P}z$ defines an injective mapping between $\mathbf{R}^{n-1}$ and $\mathbf{1}^\perp$ and, therefore, $w^T\mathcal{L}w > 0$ for all $w \in \mathbf{1}^\perp$ if and only if $z^T\mathcal{P}^T\mathcal{L}\mathcal{P}z > 0$ for all $z \in \mathbf{R}^{n-1}$. ■

Proposition 3.1 results in an equivalent convex formulation for problem (8) by

$$\begin{array}{ll} \max_{\mathcal{L}\in\mathbf{S}^n} & \gamma \\ \text{s.t.} & \mathcal{P}^T\mathcal{L}\mathcal{P} \succ \gamma\mathcal{I}_{n-1}, \end{array} \tag{9}$$

which can be solved for the optimal Laplacian matrix $\mathcal{L}_\star$ using readily available tools from semidefinite programming [79].

To obtain a set of trajectories that drive the robots from a set of initial configurations to a final configuration with associated Laplacian matrix $\mathcal{L}_\star$, the authors of [41] introduce state-dependence of the network $\mathbb{G}$ via the set of edge weights described in Fig. 2(d). Along with a set of minimum distance constraints $\|x_{ij}\|_2 \geq \rho_1$, this gives rise to the optimization problem

$$\begin{array}{ll} \max_{x\in\mathbf{R}^{dn}} & \gamma \\ \text{s.t.} & \mathcal{P}^T\mathcal{L}(x)\mathcal{P} \succ \gamma\mathcal{I}_{n-1} \\ & \|x_{ij}\|_2^2 \geq \rho_1^2, \end{array} \tag{10}$$

for all $i < j$, which now assumes a nonconvex form. Solution of problem (10) for a trajectory $x(t) \in \mathbf{R}^{dn}$ is achieved by an iterative algorithm that maximizes the algebraic connectivity at every step. For this, the distances $\|x_{ij}\|_2^2$ are differentiated and then discretized by Euler's first order method to give

$$2\left(x_i^{s+1} - x_j^{s+1}\right)^T\left(x_i^s - x_j^s\right) = [\mathcal{X}]_{ij}^{s+1} - [\mathcal{X}]_{ij}^s$$

where $\mathcal{X} \in \mathbf{R}_+^{n\times n}$ is a Euclidean distance matrix, such that $[\mathcal{X}]_{ij} = \|x_{ij}\|_2^2$ and s denotes the iteration index. Similarly, differentiating and discretizing the weights w_{ij} gives

$$w_{ij}^{s+1} = w_{ij}^s + \left.\frac{\partial f([\mathcal{X}]_{ij})}{\partial[\mathcal{X}]_{ij}}\right|_s \left([\mathcal{X}]_{ij}^{s+1} - [\mathcal{X}]_{ij}^s\right)$$

which results in a discrete Laplacian matrix $\mathcal{L}(x_s)$. Substituting in problem (10) gives

$$\begin{array}{ll} \max\limits_{x^{s+1}\in\mathbf{R}^{dn}} & \gamma \\ \text{s.t.} & \mathcal{P}^T\mathcal{L}(x^{s+1})\mathcal{P} \succ \gamma\mathcal{I}_{n-1}, \;\; [\mathcal{X}]_{ij}^{s+1} \geq \rho_1^2 \\ & 2(x_i^{s+1} - x_j^{s+1})^T(x_i^s - x_j^s) = \\ & = [\mathcal{X}]_{ij}^{s+1} - [\mathcal{X}]_{ij}^s, \end{array} \tag{11}$$

for all $i < j$. Problem (11) is essentially a linear approximation to problem (10) and, therefore, there is a potential for inconsistencies between the robot positions and their pairwise distances. This problem can be resolved if a Euclidean distance constraint is enforced on the matrix $\mathcal{X} \in \mathbf{R}_+^{n\times n}$. Such a constraint can take the form of a linear matrix inequality, which is due to the following result:

Theorem 3.2 (Euclidean distance matrix): A matrix $\mathcal{X} \in \mathbf{R}_+^{n\times n}$ is a Euclidean distance matrix if and only if $\mathcal{J}\mathcal{X}\mathcal{J} \preceq 0$ and $[\mathcal{X}]_{ii} = 0$ for all $i = 1, \ldots, n$, where $\mathcal{J} = \mathcal{I}_n - \mathbf{1}\mathbf{1}^T/n$.

Therefore, including the Euclidean distance matrix constraints from Theorem 3.2 to the maximization problem (11) ensures that there are no inconsistencies between the robot positions and the inter-robot distances. The iterative greedy algorithm proposed by the authors in [41] is guaranteed to converge as the sequence of algebraic connectivities generated by it is nondecreasing and upper bounded by $n - 1$.

B. Distributed Connectivity Maximization

A distributed solution to problem (8) can be obtained by supergradient optimization [42]. In particular, a supergradient matrix for $\lambda_2(\mathcal{L}(x))$ can be obtained by observing that

$$\lambda_2(\tilde{\mathcal{L}})z_2^Tz_2 \leq z_2^T\tilde{\mathcal{L}}z_2, \tag{12}$$

where $\tilde{\mathcal{L}} \neq \mathcal{L}$ and $z_2 \in \mathbf{1}^\perp$ is the unit eigenvector of $\mathcal{L}$ corresponding to $\lambda_2(\mathcal{L})$. The right hand side of (12) can be further expanded to give

$$\begin{aligned} z_2^T\tilde{\mathcal{L}}z_2 &= z_2^T\mathcal{L}z_2 + z_2^T(\tilde{\mathcal{L}} - \mathcal{L})z_2 \\ &= \lambda_2(\mathcal{L}) + \langle z_2z_2^T, (\tilde{\mathcal{L}} - \mathcal{L})\rangle, \end{aligned}$$

which substituted in (12) gives

$$\lambda_2(\tilde{\mathcal{L}})z_2^Tz_2 \leq \lambda_2(\mathcal{L}) + \langle z_2z_2^T, (\tilde{\mathcal{L}} - \mathcal{L})\rangle.$$

Therefore, the matrix $\mathcal{G} = z_2z_2^T$ is a supergradient for $\lambda_2(\mathcal{L})$. Then, the optimal Laplacian matrix $\mathcal{L}_\star$, can be obtained as the limit of the subgradient iteration

$$\mathcal{L}_\star^{s+1} = \mathcal{L}_\star^s + \alpha^s\mathcal{G}^s, \tag{13}$$

where $\mathcal{L}_\star$ denotes a target Laplacian matrix and s denotes the iteration index. If the step size α^s is the coefficient of a not summable but square summable series, then the supergradient method converges to the optimal value. Distributed computation of the supergradient $\mathcal{G}^s$ as well as of the eigenvectors of the Laplacian matrix $\mathcal{L}$ is discussed in [80]. According to this scheme, every robot i computes its own row of the Laplacian matrix $\mathcal{L}_\star^s$, denoted by $[\mathcal{L}_\star^s]_i$.

To obtain a set of trajectories that drive the robots from an initial configuration to a final configuration associated with the optimal Laplacian matrix $\mathcal{L}_\star = \lim_{s\to\infty}\mathcal{L}_\star^s$, the authors in [42] propose a set of distributed motion controllers $\{u_i(t)\}_{i=1}^n$ for the robots that essentially track the sequence of Laplacians $\mathcal{L}_\star^s$ generated by the supergradient algorithm (13). State-dependence of the network $\mathbb{G}$ is introduced via a set of symmetric weights that are according to Fig. 2(d) and give rise to a state-dependent Laplacian matrix $\mathcal{L}(x)$ defined by (5). Therefore, associated with every iteration of the supergradient algorithm (13) is a motion control stage, which for every robot i is captured by the following optimization problem

$$\min_{x_i\in\mathbf{R}^d} \|[\mathcal{L}(x)]_i - [\mathcal{L}_\star^s]_i\|_2^2,$$

where $[\mathcal{L}(x)]_i$ denotes the i-th row of the Laplacian matrix as a function of the robots' positions, and $[\mathcal{L}_\star^s]_i$ is the i-th row of the optimal Laplacian computed by robot i at the s-th step of the supergradient iteration. The above optimization problem is solved using potential functions and results in a controller

$$u_i(t) = -\sum_{j\in\mathbb{N}_i}\nabla_{x_i}V_{ij}(t), \tag{14}$$

for every robot i, where

$$V_{ij}(t) = \begin{cases} \left(\|x_{ij}\|_2^2 - [\mathcal{L}_\star^s]_{ij}^{-1}\right)^2, \text{if } \|x_{ij}\|_2 \leq \rho_2 \\ \left(\rho_2 - [\mathcal{L}_\star^s]_{ij}^{-1}\right)^2, \text{if } \|x_{ij}\|_2 > \rho_2 \end{cases},$$

and $[\mathcal{L}_\star^s]_{ij}^{-1}$ is the desired distance between robots i and j, given by the inverse of the (i,j)-th entry of the target Laplacian matrix. It is shown in [42] that under certain boundedness conditions on the tracking error associated with the optimal Laplacian $\mathcal{L}_\star^s$, the supergradient algorithm converges.

IV. Continuous Feedback Connectivity Control

Both approaches discussed in Section III employ discrete iterative algorithms to control the non-differentiable algebraic connectivity $\lambda_2(\mathcal{L}(x))$. However, Proposition 3.1 and the fact that the determinant of any matrix is equal to the product of its eigenvalues gives that $\lambda_2(\mathcal{L}(x)) > 0$ if and only if $\det\left(\mathcal{P}^T\mathcal{L}(x)\mathcal{P}\right) > 0$. Using this observation for state-dependent networks $\mathbb{G}$ with edge weights as in Fig. 2(b), the authors of [43] propose a class of potential fields $\phi : \mathbf{R}^{dn} \to \mathbf{R}_+$ that treat connectivity violation as an obstacle in the configuration space. This is captured in the following result.

Proposition 4.1: Define the potential function

$$\phi(x) = \log\det(\mathcal{P}^T\mathcal{L}(x)\mathcal{P})^{-1}, \tag{15}$$

Then, the closed loop system (1) with $u = -\nabla_x\phi(x)$ guarantees that $\mathbb{G}$ is connected for all time.

Proof: The proof of this result relies on positive invariance of the level sets $\phi^{-1}([0,c]) = \{x \in \mathbf{R}^{dn} \mid \phi(x) \leq c\}$ of ϕ, which is due to the fact that $\dot{\phi}(x) = -\|\nabla_x\phi(x)\|_2^2 \leq 0$. ■

Note that the potential ϕ is a convex function of the Laplacian matrix $\mathcal{L}$ [79]. However, dependence of the Laplacian on the state via the edge weights makes ϕ a nonconvex function of the $x \in \mathbf{R}^{dn}$. Therefore, even though $\det(\mathcal{P}^T\mathcal{L}(x)\mathcal{P})$ increases as a result of Proposition 4.1, $\lambda_2(\mathcal{L}(x))$ might actually decrease. This implies that the proposed control scheme ensures only local maximization of $\lambda_2(\mathcal{L}(x))$. The authors of [43] conclude by providing a closed form expression for the controller in Proposition 4.1.

Proposition 4.2: The controller $u = -\nabla_x\phi(x)$ is given by

$$u = \frac{1}{\det\mathcal{M}(x)}\begin{bmatrix} \mathrm{tr}\left[\mathcal{M}^{-1}(x)\frac{\partial}{\partial x_1}\mathcal{M}(x)\right] \\ \vdots \\ \mathrm{tr}\left[\mathcal{M}^{-1}(x)\frac{\partial}{\partial x_n}\mathcal{M}(x)\right]\end{bmatrix}, \tag{16}$$

where $\mathcal{M}(x) = \mathcal{P}^T\mathcal{L}(x)\mathcal{P}$.

Proof: Let $\mathcal{M}(x) = \mathcal{P}^T\mathcal{L}(x)\mathcal{P}$ and denote by $c_{ij}(x)$ the cofactor of the entry $m_{ij}(x)$ of the matrix $\mathcal{M}(x)$. Let $\mathcal{C}(x)$ denote the cofactor matrix and denote by $c_{ij}^T(x)$ the (i,j)-th entry of $\mathcal{C}^T(x)$, i.e., $c_{ij}^T(x) = c_{ji}(x)$. Since the determinant is a differentiable function of matrix entries, in particular it is a sum of products of entries, the chain rule gives,

$$\frac{\partial}{\partial x_k}\det\mathcal{M}(x) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left(\frac{\partial}{\partial m_{ij}}\det\mathcal{M}(x)\right)\frac{\partial}{\partial x_k}m_{ij}(x)$$

For all $j = 1,\dots,n-1$, computation of the Laplace expansion of the determinant along the j-th column gives $\det\mathcal{M}(x) = \sum_{i=1}^{n-1}c_{ij}(x)m_{ij}(x)$ and hence $\frac{\partial}{\partial m_{ij}}\det\mathcal{M}(x) = c_{ij}(x)$. Therefore,

$$\begin{aligned}\frac{\partial}{\partial x_k}\det\mathcal{M}(x) &= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}c_{ij}(x)\frac{\partial}{\partial x_k}m_{ij}(x)\\ &= \sum_{j=1}^{n-1}\sum_{i=1}^{n-1}c_{ji}^T(x)\frac{\partial}{\partial x_k}m_{ij}(x) = \mathrm{tr}\left[\mathcal{C}^T(x)\frac{\partial}{\partial x_k}\mathcal{M}(x)\right]\end{aligned}$$

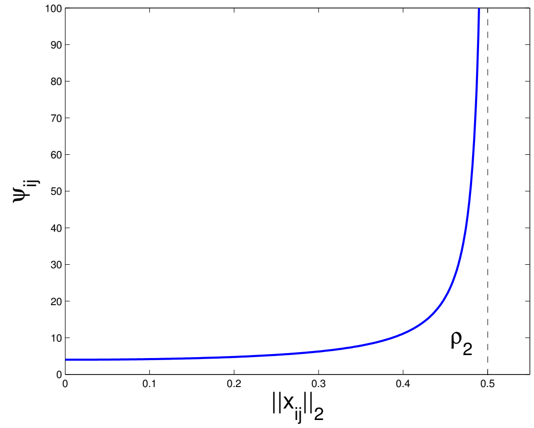


Fig. 3. The artificial potential function $\psi_{ij}(\|x_{ij}\|_2)$. The function is symmetric with respect to x_i and x_j, and when bounded, it guarantees edge preservation for $\|x_{ij}\|_2 \to \rho_2$. Here, the function is plotted for $\rho_2 = 0.5$.

A direct consequence of the Laplace expansion of the determinant is the identity $I \cdot \det\mathcal{M}(x) = \mathcal{M}(x)\mathcal{C}^T(x)$. Proposition 4.1 guarantees that $\lambda_2(\mathcal{L}(x)) > 0$ for all time, and so $\det\mathcal{M}(x) > 0$ for all $x \in \mathbf{R}^{dn}$. Thus $\mathcal{M}(x)$ is always positive definite, and hence invertible. Therefore, by left multiplication of the previous identity by $\mathcal{M}^{-1}(x)$, we get $\mathcal{M}^{-1}(x)\cdot\det\mathcal{M}(x) = \mathcal{C}^T(x)$, and substituting in the expression for $\frac{\partial}{\partial x_k}\det\mathcal{M}(x)$ we get,

$$\begin{aligned}\frac{\partial}{\partial x_k}\det\mathcal{M}(x) &= \mathrm{tr}\left[\det\mathcal{M}(x)\cdot\mathcal{M}^{-1}(x)\frac{\partial}{\partial x_k}\mathcal{M}(x)\right]\\ &= \det\mathcal{M}(x)\cdot\mathrm{tr}\left[\mathcal{M}^{-1}(x)\frac{\partial}{\partial x_k}\mathcal{M}(x)\right]\end{aligned}$$

where $\mathcal{M}^{-1}(x) = \left(\mathcal{P}^T\mathcal{L}(x)\mathcal{P}\right)^{-1}$ and $\frac{\partial}{\partial x_k}\mathcal{M}(x) = \mathcal{P}^T\frac{\partial}{\partial x_k}\mathcal{L}(x)\mathcal{P}$, and the result follows directly from Proposition 4.1 and a simple application of the chain rule. ■

V. Hybrid Feedback Connectivity Control

The approach discussed in Section IV is centralized since every robot requires knowledge of the whole network structure captured by $\mathcal{L}(x)$ to compute its controller (Proposition 4.2). The key idea employed in [56] and [46] to regulate the structure of the proximity-based network $\mathbb{G}$ in a distributed fashion is the introduction of a binary control signal $\sigma \in \{0,1\}^{n\times n}$, such that

$$[\sigma]_{ij} = \begin{cases}1, & \text{to activate the link } (i,j) \in \vec{\mathbb{E}}\\ 0, & \text{to deactivate the link } (i,j) \in \vec{\mathbb{E}}\end{cases}.$$

This gives rise to the weighted graph $\mathbb{G}_\sigma = (\mathbb{V}, \mathbb{W}_\sigma)$ where $\mathbb{W}_\sigma : \mathbb{V}\times\mathbb{V}\times\mathbf{R}_+ \to \mathbf{R}_+$ is the set of edge weights such that

$$\mathbb{W}_\sigma(i,j,t) = w_{ij}^\sigma(t)$$

with $w_{ij}^\sigma = w_{ij}[\sigma]_{ij}$, for w_{ij} given by (3). Therefore, the control signal σ is essentially a discrete switch on the links of the network $\mathbb{G}$, but only affects existing links for which $w_{ij} > 0$. The edge and neighbor sets associated with the graph $\mathbb{G}_\sigma$ are defined by $\vec{\mathbb{E}}_\sigma = \{(i,j) \mid w_{ij}^\sigma > 0\}$ and

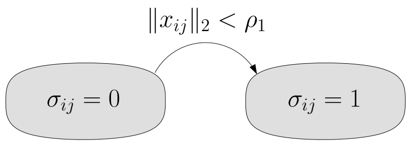


Fig. 4. Hysteresis protocol for adding interagent energy functions to the total energy function only when agents get within a distance ρ_1 of each other, rather than when they first encounter each other at a distance ρ_2.

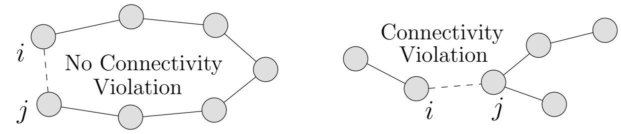


Fig. 5. Control challenges requiring knowledge of the network structure. Without such knowledge, deletion of a link (i,j) can either violate connectivity (right) or not (left).

$\mathbb{N}_i^\sigma = \{j \in \mathbb{V} \mid (i,j) \in \vec{\mathbb{E}}_\sigma\}$, respectively. Based on this idea, the authors of [56] and [46] propose a hybrid model for the mobile network $\mathbb{G}$ consisting of single integrator robots (1) and controllers given by

$$u_i^\sigma = -\sum_{j\in\mathbb{N}_i^\sigma} \nabla_{x_i}\psi_{ij}. \tag{17}$$

The functions $\psi_{ij} : \mathbf{R}_+ \to \mathbf{R}_+$ are artificial potential functions defined on the links of the network, which in the case of connectivity control take the form (Fig. 3)

$$\psi_{ij} = \frac{1}{\rho_2^2 - \|x_{ij}\|_2^2}, \tag{18}$$

to ensure link preservation between adjacent robots. The rest of this section discusses two particular choices for the control signal σ that ensure connectivity of the mobile network $\mathbb{G}$.

A. Maintaining Communication Links

The approach followed in [56] relies on maintaining and increasing the number of links in the network. Since

$$\lim_{\|x_{ij}\|_2 \to \rho_2^-} \psi_{ij} = \infty,$$

infinite energies ψ_{ij} take place in the control laws (17) when two robots i and j form an edge between them, i.e., when they move within distance ρ_2 of each other. To address this problem, the authors of [56] introduce a hysteresis into the system through the signal σ given by the state machine in Fig. 4. In particular, the signal $[\sigma]_{ij}$ is such that the total energy is affected by an edge (i,j) that was previously not contributing to the total energy only when $\|x_{ij}\|_2 < \rho_1$, where $0 < \rho_1 < \rho_2$ is the predefined switching threshold that regulates how fast inter-robot information is included in the control law. Once the edge is allowed to contribute to the total energy, it keeps doing so for all subsequent times. In particular, the signal $[\sigma]_{ij}$ is defined by

$$[\sigma]_{ij}(t^+) = \begin{cases} 0, & \text{if } [\sigma]_{ij}(t^-) = 0 \text{ and } \|x_{ij}\|_2 \geq \rho_1 \\ 1, & \text{otherwise} \end{cases},$$

where the notation $[\sigma]_{ij}(t^+)$ and $[\sigma]_{ij}(t^-)$ denotes the value of $[\sigma]_{ij}$ before and after the state transition in Fig. 4. It can be shown that this control scheme maintains all links in $\mathbb{G}_\sigma$ and, therefore, ensures connectivity of the network [46], [56].

Proposition 5.1: Consider the closed loop system (1)–(17). Then, all links in $\mathbb{G}_\sigma$ are maintained.

Proof: Let

$$\psi_\sigma = \frac{1}{2}\sum_{i=1}^n \psi_i^\sigma,$$

where $\psi_i^\sigma = \sum_{j\in\mathbb{N}_i^\sigma}\psi_{ij}$, denote the total energy of the system and observe that

$$\begin{aligned} \frac{1}{2}\sum_{i=1}^n \dot{\psi}_i^\sigma &= \frac{1}{2}\sum_{i=1}^n\sum_{j\in\mathbb{N}_i^\sigma} \dot{x}_{ij}^T\nabla_{x_{ij}}\psi_{ij} \\ &= \frac{1}{2}\sum_{i=1}^n\sum_{j\in\mathbb{N}_i^\sigma} \left(\dot{x}_i^T\nabla_{x_{ij}}\psi_{ij} - \dot{x}_j^T\nabla_{x_{ij}}\psi_{ij}\right) \\ &= \frac{1}{2}\sum_{i=1}^n\sum_{j\in\mathbb{N}_i^\sigma} \left(\dot{x}_i^T\nabla_{x_i}\psi_{ij} + \dot{x}_j^T\nabla_{x_j}\psi_{ij}\right) \\ &= \sum_{i=1}^n\sum_{j\in\mathbb{N}_i^\sigma} \dot{x}_i^T\nabla_{x_i}\psi_{ij} = \sum_{i=1}^n \dot{x}_i^T\nabla_{x_i}\psi_i^\sigma \end{aligned}$$

by symmetry of the functions ψ_{ij}. Therefore,

$$\dot{\psi}_\sigma = -\sum_{i=1}^n \|\nabla_{x_i}\psi_i^\sigma\|_2^2 \leq 0,$$

which implies that the level sets $\psi_\sigma^{-1}([0,c])$ of ψ_σ are positively invariant and, hence, no links are lost. ■

B. Incorporating Link Deletions

The approach followed in [46] extends the hysteresis model for link activations introduced in [56] to also account for connectivity preserving link deactivations. For this, the authors in [46] propose a set of control signals $\{\sigma_i\}_{i=1}^n$, where $\sigma_i \in \{0,1\}^{n\times n}$ denotes the signal associated with robot i, that give rise to local neighbor sets $\mathbb{N}_i^{\sigma_i}$ defined as in Section V.

The key idea behind the approach developed in [46] is to employ distributed consensus to populate the signals σ_i with non-adjacent active links and then use these signals to check link deactivations with respect to connectivity (Fig. 5). The latter objective is possible since connectivity verification does not require the actual edge weights, but only knowledge of what links in the network are active, which is captured by the signals σ_i. In other words, the signals σ_i can be thought of as an abstraction of the adjacency matrices of the graphs $\mathbb{G}_{\sigma_i}$ obtained when the signals σ_i are applied to $\mathbb{G}$. The proposed update rule is[2]

$$\sigma_i(s+1) = \neg(\sigma_i(s) \leftrightarrow \omega_i(s)), \tag{19}$$

where $\omega_i \in \{0,1\}^{n\times n}$ is such that $[\omega_i]_{jk} = 1$ if a control action is taken to activate or deactivate link (j,k) (Table I).

[2]The symbols $\neg$, $\wedge$, $\vee$, $\to$ and $\leftrightarrow$ stand for the boolean operators NOT, AND, OR, IF, THEN and IF AND ONLY IF, respectively (in the case of matrices, they are applied elementwise on their entries). The discrete time semantics in (19) are associated with discrete communication time instances between adjacent robots.

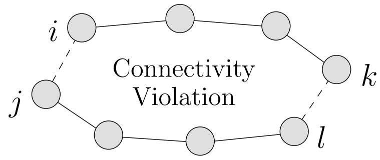


Fig. 6. Control challenges due to multiple link deletions. In the absence of an agreement protocol, simultaneous deletion of links (i, j) and (k, l) violates connectivity.

TABLE I
LINK DYNAMICS

$[\sigma_i(s)]_{jk}$	$[\omega_i(s)]_{jk}$	$[\sigma_i(s+1)]_{jk}$
1	1	0
1	0	1
0	1	1
0	0	0

It is shown in [46] that ω_i can be decomposed into two disjoint components ω_i^a and ω_i^d regulating activations and deactivations, respectively, as

$$\omega_i = \left(\left(\underbrace{\left(\neg\sigma_i \wedge (\vee_{j\in\mathbb{N}_i^{\sigma_i}}\sigma_j)\right)}_{\mathrm{I}} \vee \underbrace{(\neg\sigma_i \wedge \eta_i)}_{\mathrm{II}}\right) \wedge \omega_i^a\right) \vee \left(\sigma_i \wedge \omega_i^d\right),$$

where

- the (k,l)-th entry of Term I (with $k, l \neq i$) is equal to 1 if there exists an active link between robots k and l that is known to robot i's neighbors, i.e., $\left[\vee_{j\in\mathbb{N}_i^{\sigma_i}}\sigma_j\right]_{kl} = 1$, but is not known to robot i, i.e., $[\neg\sigma_i]_{kl} = 1$,
- the (k,l)-th entry of Term II with $k = i$ or $l = i$ is equal to 1 if there does not exist an active link between robots k and l, i.e., $[\neg\sigma_i]_{kl} = 1$, and is always zero if $k, l \neq i$.

The condition that $k = i$ or $l = i$ in Term II is captured by the matrix $\eta_i = \vee_{j\neq i}(e_i e_j^T \vee e_j e_i^T)$, where e_i is an $n \times 1$ column vector with all entries 0 except for the i-th entry that is 1. Clearly, if the (k,l)-th entry of either Term I or Term II is equal to 1, then this entry indicates a link that can possibly become activated if the activation control action becomes $[\omega_i^a]_{kl} = 1$. Similarly, the control action ω_i^d can only deactivate links (k,l) that robot i considers active, i.e., $[\sigma_i]_{kl} = 1$. It is shown in [46] that the dynamics (19) resemble a consensus algorithm with inputs on the control signals σ_i that in the case of no inputs, i.e., if $\omega_i^a = 1_{n\times n}$ and $\omega_i^d = 0_{n\times n}$, reduce to the usual consensus update

$$\sigma_i := \vee_{j\in\mathbb{N}_i^{\sigma_i}} (\sigma_i \vee \sigma_j).$$

as desired. The choice of the control actions ω_i^a and ω_i^d needs to satisfy the following two conditions:

(a) σ_i is updated with all active links in $\mathbb{G}_\sigma$ and,
(b) connectivity of $\mathbb{G}_\sigma$ is not violated by link deactivations.

Condition (a) is satisfied by the link addition controller

$$\omega_i^a = \underbrace{(\neg\eta_i)}_{\mathrm{III}} \vee \underbrace{\left(\eta_i \wedge \left(\mathcal{X} < \frac{\rho_1^2}{n}\mathbf{1}\mathbf{1}^T\right)\right)}_{\mathrm{IV}}, \tag{20}$$

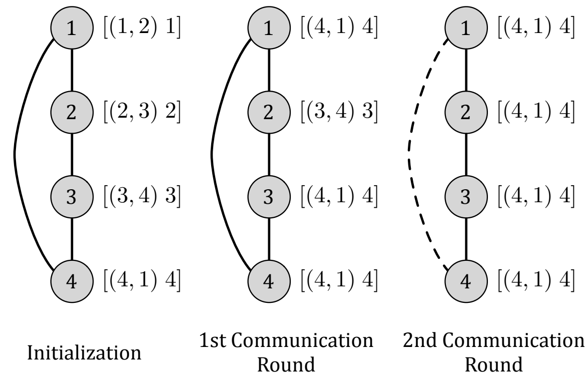


Fig. 7. An example of a link deactivation auction taking place in a network of 4 robots. Next to every robot in brackets is shown its deletion request $r_i = [(i, j)\ b_i]$ containing a desired link (i, j) with $j \in \mathbb{S}_i$ that if deactivated does not violate connectivity, and the associated bid b_i. Initialization is as shown in the network at the left. During the first communication round, robot 1 compares its bid $b_1 = 1$ with the bids of its neighbors, $b_4 = 4$ and $b_2 = 2$. Since among its neighbors, robot 4 has placed the highest bid, robot 1 updates its request with the request of robot 4, i.e., $r_1 = [(4, 1)\ 4]$ (cf. (21)). Similar updates take place for the requests of the other robots. After two communication rounds, all robots have agreed on the request with the highest bid $[(4, 1)\ 4]$. Then, robot 4 physically deactivates (dashed line) the link $(4, 1)$ and along with all other robots updates its signal σ_i (cf. (22)).

where

- Term III ensures that $[\omega_i^a]_{kl} = 1$ whenever $k, l \neq i$, i.e., that all active links in the network known to robot i's neighbors will be activated in σ_i as well, and
- Term IV ensures that $[\omega_i^a]_{kl} = 1$ whenever $k = i$ or $l = i$ and the distance $[\mathcal{X}]_{kl} = \|x_{kl}\|_2^2$ between robots k and l (with $k = i$ or $l = i$) is lower than the link activation threshold ρ_1, i.e., that links between robot i and close-by agents will be activated ($\mathcal{X} \in \mathbf{R}_+^{n\times n}$ denotes a Euclidean distance matrix).

Condition (b) needs to address the fact that simultaneous link deactivations by multiple non-adjacent robots may disconnect $\mathbb{G}_\sigma$ (Fig. 6). For this, the authors in [46] propose a market-based framework to achieve agreement of all robots on one single link deactivation as the outcome of every auction. In particular, every robot i selects a neighbor j in the set

$$\mathbb{S}_i = \left\{k \in \mathbb{N}_i^{\sigma_i} \mid \|x_{ik}\|_2 \in [\rho_1, \rho_2),\ \lambda_2(\mathcal{L}_i^{-k}) > 0\right\},$$

where $\mathcal{L}_i^{-k}$ is the Laplacian matrix of the network $\mathbb{G}_{\sigma_i}$ minus the link (i,k), such that if the link (i,j) with $j \in \mathbb{S}_i$ is deactivated, then the network $\mathbb{G}_{\sigma_i}$ remains connected. The rest of the algorithm relies on multi-hop propagation of deletion requests $r_i = [r_{i1}\ r_{i2}\ r_{i3}]^T \in \mathbf{R}^3$ containing the requested link $(r_{i1}, r_{i2}) \in \vec{\mathbb{E}}$ and an associated bid $r_{i3} \in \mathbf{R}_+$, such that initially $r_{i1} = i$ and $r_{i2} = j \in \mathbb{S}_i$ for all robots i. With every communication round, request r_i is updated with the request r_j corresponding to the robot j that has placed the highest bid r_{j3}, i.e.,

$$r_i = r_j \quad \text{with} \quad j \in \mathrm{argmax}_{k\in\mathbb{N}_i^{\sigma_i}}\{r_{i3}, r_{k3}\}, \tag{21}$$

and employs a "maximum label" rule to break ties. Note that (21) is essentially a maximum consensus update on the bids r_{i3} and will converge to a common outcome r_i for all robots

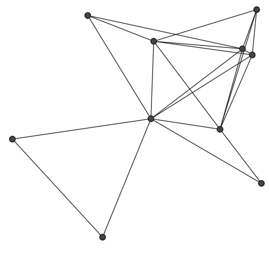


Fig. 8. Execution of the rendezvous control strategy until the graph is a complete graph.

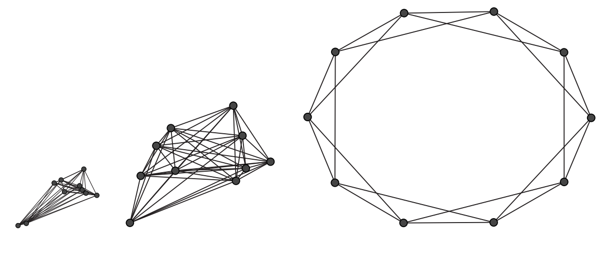


Fig. 9. Illustration of how the complete graph is changed to the desired formation using only local information.

when all bids have been compared to each other. If at least one robot has placed a positive bid, i.e., if $r_{i3} > 0$, then the controller

$$\omega_i^d = e_{r_{i1}} e_{r_{i2}}^T \vee e_{r_{i2}} e_{r_{i1}}^T, \tag{22}$$

deactivates the link (r_{i1}, r_{i2}) from $\mathbb{G}_{\sigma_i}$, and the process is repeated for a new link deactivation (Fig. 7). If $r_{i3} = 0$, then $\omega_i^d = 0_{n\times n}$, i.e., no link is deactivated form $\mathbb{G}_{\sigma_i}$.

Communication time delays, packet losses, and the asymmetric network structure, may result in auctions starting asynchronously, outdated information being used for future decisions, and consequently, robots reaching different decisions for the same auction. In the absence of a common global clock, the authors of [46] propose an event triggered synchronization scheme, where a triggering event corresponds to receipt of a communication message, that ensures that "fast" robots wait for their "slower" peers to reach a decision too. Alltogether, this framework gives rise to the following result.

Theorem 5.2 (Connectivity maintenance): Assume that the network $\mathbb{G}_\sigma$ is initially connected. Then, the closed loop system (19)–(20)–(22) guarantees that $\mathbb{G}_\sigma$ remains connected for all time.

Proof: Assume that the local networks $\mathbb{G}_{\sigma_i}$ are initialized with nearest neighbor links only. Then, the proof relies on the following observations:

(a) All network estimates $\mathbb{G}_{\sigma_i}$ are spanning subgraphs of the overall network $\mathbb{G}_\sigma$, which implies that connectivity can be checked locally for $\mathbb{G}_{\sigma_i}$ and then extended to $\mathbb{G}_\sigma$.
(b) The market-based maximum consensus (21) ensures agreement of all robots on the deactivation request which, therefore, does not violate connectivity.
(c) Synchronization ensures that no outdated information is used in (21).

Consequently, links can be deactivated continuously one-by-one, without violating connectivity of the network. ∎

VI. Applications of Connectivity Control

A. Connectivity Preserving Rendezvous

A canonical example in which connectivity maintenance is crucial is the so-called rendezvous problem. Here, the robots are required to meet at a common, not a priori specified location without relying on global positioning. Instead, the only information available to them is the relative displacement, i.e., robot i, at position x_i, has access to $x_j - x_i$ if i and j are neighbors, i.e., if they are within sensing range of each other. A linear control strategy that achieves this objective is

$$\dot{x}_i = \sum_{j\in\mathbb{N}_i} (x_j - x_i) \tag{23}$$

as long as the graph $\mathbb{G}$ is connected for all times. However, as shown in [56], initially connected proximity networks that evolve according to (23) are not guaranteed to remain connected throughout time. Instead, nonlinear coordination models are needed, and one that achieves rendezvous while ensuring connectivity (Problem 1), is the hybrid control strategy under consideration in Proposition 5.1. In particular, the model employed in [56] is

$$\dot{x}_i = \sum_{j\in\mathbb{N}_i^\sigma} \frac{2\rho_2^2}{(\rho_2^2 - \|x_{ij}\|_2^2)^2}(x_j - x_i), \tag{24}$$

which, not only ensures that no edges are lost, but it also achieves rendezvous in the sense that all agents asymptotically approach the same location. This is due to the modified potentials $\bar{\psi}_{ij} = \frac{\|x_{ij}\|_2^2}{\rho_2^2 - \|x_{ij}\|_2^2}$ in (24) that along with link maintenance (Section V-A) also capture the rendezvous objective. An example of this behavior is shown in Fig. 8.

It should be noted that the rendezvous control law often serves a cohesion purpose, i.e., ensures that the robots in the team stay close together. Nevertheless, exact rendezvous is not necessarily a good thing and a reactive, collision-avoidance controller could be added to the control strategy to avoid overlapping of the actual robots.

B. Connectivity Preserving Formation Control

A variation to the rendezvous objective is the problem of driving the robots to a desired target configuration, rather than to a common target location. We assume that this target configuration can be encoded through $\zeta_1, \ldots, \zeta_n \in \mathbf{R}^d$, with the interpretation that agent i should go to location ζ_i, for $i = 1, \ldots, n$. Since formations are considered rotationally and translationally invariant objects in the configuration space, their exact location is not of interest . Therefore, the formation control objective is to achieve

$$x_i = \zeta_i + \tau, \quad \forall\, i = 1, \ldots, n$$

for some constant $\tau \in \mathbf{R}^d$. In other words, τ corresponds to the constant offset from the target configuration that the agents should agree on. But, by letting $\chi_i = x_i - \zeta_i$, and

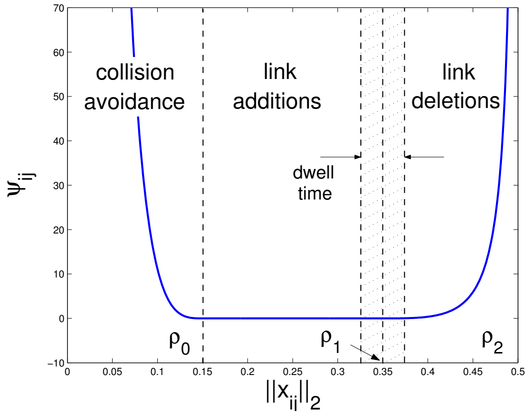


Fig. 10. The artificial potential function $\psi_{ij}(\|x_{ij}\|_2)$. The function is symmetric with respect to x_i and x_j, and when bounded, it guarantees both collision avoidance for $\|x_{ij}\|_2 \to 0$ and edge preservation for $\|x_{ij}\|_2 \to \rho_2$. Here, the function is plotted for $\rho_0 = 0.15$, $\rho_1 = .35$ and $\rho_2 = 0.5$. The dwell time at the switching threshold ρ_2 ensures that the resulting switched system is well defined [64].

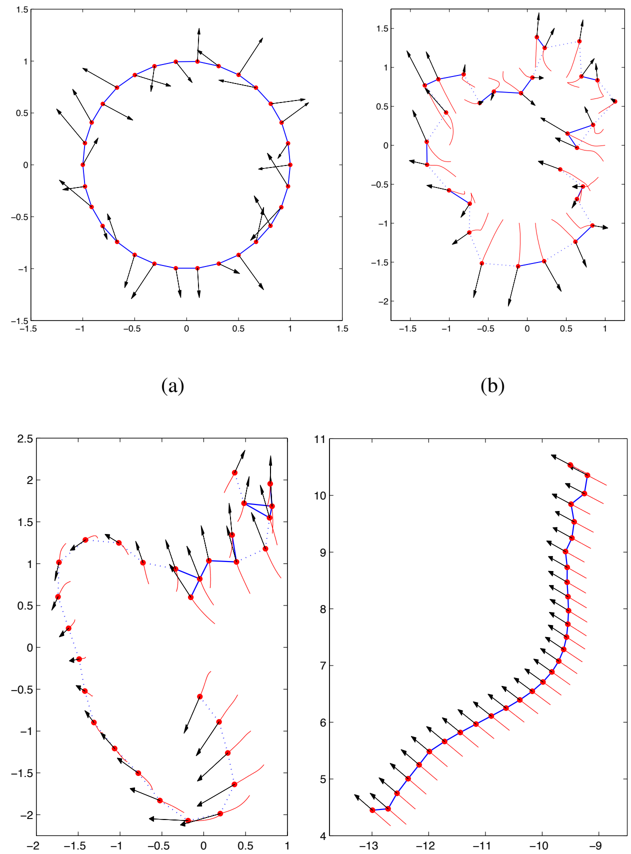
running the connectivity preserving rendezvous algorithm (24) over the χ_i's instead of over the x_i's, it is ensured that the offsets χ_i reach a common value which corresponds directly to the offset τ [81]. Note that since $\dot{x}_i = \dot{\chi}_i$, this strategy directly gives desired motions for the robots in terms of their velocities. Moreover, all that is needed to compute these control laws are the relative displacements $x_i - x_j$ between neighboring robots as well as the desired predefined relative displacement $\zeta_i - \zeta_j$. This is highlighted in Fig. 9.

C. Connectivity Preserving Flocking

Flocking has been given many definitions and various models have been proposed so far [82]–[85]. Therefore, it is understood quite differently by different authors. In this paper we focus on the model proposed by Reynolds, developed to simulate social aggregation phenomena, such as flocks of birds and schools of fish [86]. Reynolds called the generic simulated flocking creatures "boids" and developed his flocking model based on three simple steering behaviors that describe how an individual robot maneuvers given the positions and velocities of its nearby flockmates:

- Alignment: Steer towards the average heading of local flockmates.
- Separation: Steer to avoid crowding of local flockmates.
- Cohesion: Steer towards the average position of local flockmates.

In Reynolds model, every robot has access to the whole scenes geometric description, however, flocking requires information from nearest neighbor flockmates only. This neighborhood depends on a distance and an angle from the robots direction of motion, and can be thought of as model of limited perception (such as fish in murky water) or as the region where a robot's motion is influenced by its flockmates. Superposition of these three rules results in all robots moving as a flock while avoiding collisions. Inspired by Reynolds model, the authors of [87] proposed local control laws that allow a team of robots



(a) (b) (c) (d)

Fig. 11. Connectivity preserving flocking of $n = 30$ robots for a sparse initial configuration where connectivity can not be trivially maintained. It can be seen that the network remains connected while all robot velocities are asymptotically aligned. Dotted lines indicate communication links that are candidates for deletion (Fig. 10).

with double integrator dynamics

$$\dot{x}_i = v_i \tag{25a}$$

$$\dot{v}_i = -\sum_{j\in\mathbb{N}_i}(v_i - v_j) - \sum_{j\in\mathbb{N}_i}\bar{\psi}_{ij} \tag{25b}$$

to align their velocities, move with a common speed and achieve desired inter-robot distances while avoiding collisions with each other. Stability results were obtained using non-smooth analysis and algebraic graph theory and critically relied on connectivity of the communication network. Based on these results, the authors of [64] proposed integration of the dynamics (25) with the connectivity control framework developed in Section V-B and the artificial potentials (Fig. 10)

$$\bar{\psi}_{ij} = \begin{cases} \frac{1}{\|x_{ij}\|_2^2} + P_1(\|x_{ij}\|_2), & \|x_{ij}\|_2 \in (0, \rho_0] \\ 0, & \|x_{ij}\|_2 \in (\rho_0, \rho_1) \\ \frac{1}{\rho_2^2 - \|x_{ij}\|_2^2} + P_2(\|x_{ij}\|_2), & \|x_{ij}\|_2 \in [\rho_1, \rho_2) \end{cases}, \tag{26}$$

with $0 < \rho_0 < \rho_1 < \rho_2$ and $P_k(\|x_{ij}\|_2) \triangleq a_k\|x_{ij}\|_2^2 + b_k\|x_{ij}\|_2 + c_k$ for $k = 1, 2$ such that $\psi_{ij} \in \mathbf{C}^2$ in $(0, \rho_2)$. The resulting multi-robot hybrid system was shown to guarantee the flocking behavior of the team while preserving connectivity of the network (Fig. 11).

VII. Conclusions

In this paper we provided a theoretical framework for controlling graph connectivity in mobile robot networks. We presented a cohesive overview of the key results in [41]–[43], [46], [56] and discussed basic notions of network connectivity as well as control theoretic methods for connectivity preservation. These methods relied on a variety of mathematical tools, ranging from spectral graph theory and semidefinite programming to maximize the algebraic connectivity of a network, to gradient descent algorithms and hybrid systems to ensure topology control in a least restrictive manner. We also discussed applications of connectivity control to multi-robot rendezvous [56], flocking [64] and formation control [56], where so far, network connectivity had been considered an assumption. A byproduct of this work was to classify the available literature with respect to the connectivity metrics and solution techniques and provide a reference for future research.

References

[1] J. Cortes, "Achieving coordination tasks in finite time via nonsmooth gradient flows," in *Proc. 44th IEEE Conference on Decision and Control and European Control Conference*, Seville, Spain, December 2005, pp. 6376–6381.

[2] ——, "Distributed algorithms for reaching consensus on general functions," *Automatica*, vol. 44, no. 3, pp. 726–737, 2008.

[3] R. Olfati-Saber and R. M. Murray, "Decentralized control of connectivity for multi-agent systems," in *Proc. 2003 American Control Conference*, 2003, pp. 951–956.

[4] ——, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, Sptember 2004.

[5] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1465–1476, Sptember 2004.

[6] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, June 2003.

[7] L. Moreau, "Stability of multi-agent systems with time-dependent communication links," *IEEE Transactions on Automatic Control*, vol. 50, no. 2, pp. 169–182, February 2005.

[8] W. Ren and R. W. Beard, "Consensus seeking in multi-agent systems under dynamically changing interaction topologies," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 655–661, May 2005.

[9] Z. Lin, M. Brouke, and B. Francis, "Local control strategies for groups of mobile autonomous agents," *IEEE Transactions on Automatic Control*, vol. 49, no. 4, pp. 622–629, April 2004.

[10] V. Blondel, J. M. Hendrickx, A. Olshevsky, and J. N. Tsitsiklis, "Convergence in multi-agent coordination, consensus, and flocking," in *Proc. 44th IEEE Conference on Decision and Control and European Control Conference*, Seville, Spain, December 2005, pp. 2996–3000.

[11] D. Bauso, L. Giarre, and R. Pesenti, "Nonlinear protocols for optimal distributed consensus in networks of dynamic agents," *Systems and Control Letters*, vol. 55, no. 11, pp. 918–928, 2006.

[12] M. Mehyar, D. Spanos, J. Pongsjapan, S. H. Low, and R. M. Murray, "Achieving coordination tasks in finite time via nonsmooth gradient flows," in *Proc. 44th IEEE Conference on Decision and Control and European Control Conference*, Seville, Spain, December 2005, pp. 7446–7451.

[13] P.-A. Bliman and G. Ferrari-Trecate, "Average consensus problems in networks of agents with delayed communications," in *Proc. 44th IEEE Conference on Decision and Control and European Control Conference*, Seville, Spain, December 2005, pp. 7066–7071.

[14] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. of the IEEE*, vol. 95, no. 1, pp. 215–233, January 2007.

[15] A. Neskovic, N. Neskovic, and G. Paunovic, "Modern approaches in modeling of mobile radio systems propagation environment," *IEEE Communications Surveys*, vol. 3, no. 3, pp. 1–12, 2000.

[16] J. D. Parsons, *The Mobile Radio Propagation Channel*. John Willey & Sons, 2000.

[17] K. Pahlavan and A. H. Levesque, *Wireless Information Networks*. New York, NY: John Willey & Sons, 1995.

[18] A. J. Goldsmith and S. B. Wicker, "Design challenges for energy-constrained ad hoc wireless networks," *IEEE Wireless Communications Magazine*, vol. 9, no. 4, pp. 8–27, August 2002.

[19] P. Gupta and P. R. Kumar, "Critical power for asymptotic connectivity," in *Proc. 37th IEEE Conference on Decision and Control*, Tampa, FL, December 1998, pp. 1106–1110.

[20] B. Chen, K. Jamieson, H. Balakrishnan, and R. Morris, "Span: An energy-efficient coordination algorithm for topology maintenance in ad hoc wireless networks," *Wireless Networks*, vol. 8, no. 5, pp. 481–494, September 2002.

[21] R. Ramanathan and R. Rosales-Hain, "Topology control of multihop wireless networks using transmit poweradjustment," in *Proc. 19th Annual Joint Conference of the IEEE Computer and Communications Societies (INFOCOM)*, Tel Aviv, Israel, 2000, pp. 404–413.

[22] E. L. Lloyd, R. Liu, M. V. Marathe, R. Ramanathan, and S. S. Ravi, "Algorithmic aspects of topology control problems for ad hoc networks," *Mobile Networks and Applications*, vol. 10, no. 1-2, pp. 19–34, February 2005.

[23] L. Ning, J. C. Hou, and L. Sha, "Design and analysis of an mst-based topology control algorithm," *IEEE Transactions on Wireless Communications*, vol. 4, no. 3, pp. 1195–1206, May 2005.

[24] R. D'Souza, D. Galvin, C. Moore, and D. Randall, "Global connectivity from local geometric constraints for sensor networks with various wireless footprints," in *Proc. 5th International Conference on Information Processing in Sensor Networks*, Nashville, TN, April 2006, pp. 19–26.

[25] L. Li, J. Y. Halpern, P. Bahl, Y. M. Wang, and R. Wattenhofer, "A cone-based distributed topology-control algorithm for wireless multi-hop networks," *IEEE/ACM Transactions on Networking*, vol. 13, no. 1, pp. 147–159, February 2005.

[26] S. Poduri, S. Pattem, B. Krishnamachari, and G. S. Sukhatme, "Using local geometry for tunable topology control in sensor networks," *IEEE Transactions on Mobile Computing*, vol. 8, no. 2, pp. 218–230, February 2009.

[27] R. Wattenhofer and A. Zollinger, "Xtc: A practical topology control algorithm for ad-hoc networks," in *Proc. 18th IEEE International Parallel and Distributed Processing Symposium*, Santa Fe, NM, April 2004, pp. 216–223.

[28] M. M. Zavlanos, A. Tahbaz-Salehi, A. Jadbabaie, and G. J. Pappas, "Distributed topology control of dynamic networks," in *Proc. 2008 American Control Conference*, Seattle, WA, June 2008, pp. 2660–2665.

[29] F. Xue and P. R. Kumar, "The number of neighbors needed for connectivity of wireless networks," *Wireless Networks*, vol. 10, no. 2, pp. 169–181, March 2004.

[30] O. Dousse, F. Baccelli, and P. Thiran, "Impact of interferences on connectivity in ad hoc networks," *IEEE/ACM Transactions on Networking*, vol. 13, no. 2, pp. 425–436, 2005.

[31] M. Fiedler, "Absolute algebraic connectivity of trees," *Linear and Multilinear Algebra*, vol. 26, pp. 85–106, 1990.

[32] Y. Wan, S. Roy, X. Wang, A. Saberi, T. Yang, M. Xue, and B. Malek, "On the structure of graph edge designs that optimize the algebraic connectivity," in *Proc. 47th IEEE Conference on Decision and Control*, Cancun, Mexico, December 2008, pp. 805–810.

[33] A. Ghosh and S. Boyd, "Growing well-connected graphs," in *Proc. 45th IEEE Conference on Decision and Control*, San Diego, CA, December 2006, pp. 6605–6611.

[34] S. Boyd, "Convex optimization of graph laplacian eigenvalues," in *Proc. International Congress of Mathematicians*, Madrid, Spain, August 2006, pp. 1311–1319.

[35] M. Mesbahi, "On state-dependent dynamic graphs and their controllability properties," *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 387–392, March 2005.

[36] H. Ando, Y. Oasa, I. Suzuki, and M. Yamashita, "Distributed memoryless point convergence algorithm for mobile robots with limited visibility," *IEEE Transactions on Robotics and Automation*, vol. 15, no. 5, pp. 818–828, October 1999.

[37] M. Fiedler, "Algebraic connectivity of graphs," *Czechoslovak Mathematical Journal*, vol. 23, no. 98, pp. 298–305, 1973.

[38] B. Mohar, "The laplacian spectrum of graphs," in *Graph Theory, Combinatorics, and Applications*, Y. Alavi, G. Chartrand, O. Ollermann, and A. Schwenk, Eds. New York, NY: Wiley, 1991, pp. 871–898.

[39] R. Merris, "Laplacian matrices of a graph: A survey," *Linear Algebra its Applications*, vol. 197, pp. 143–176, 1994.

[40] C. Godsil and G. Royle, *Algebraic Graph Theory*, ser. Graduate Texts in Mathematics. Berlin, Germany: Springer-Verlag, 2001, vol. 207.

[41] Y. Kim and M. Mesbahi, "On maximizing the second smallest eigenvalue of a state-dependent graph laplacian," *IEEE Transactions on Automatic Control*, vol. 51, no. 1, pp. 116–120, January 2006.

[42] M. C. DeGennaro and A. Jadbabaie, "Decentralized control of connectivity for multi-agent systems," in *Proc. 45th IEEE Conference on Decision and Control*, San Diego, CA, December 2006, pp. 3628–3633.

[43] M. M. Zavlanos and G. J. Pappas, "Potential fields for maintaining connectivity of mobile networks," *IEEE Transactions on Robotics*, vol. 23, no. 4, pp. 812–816, August 2007.

[44] P. Yang, R. Freeman, G. Gordon, K. Lynch, S. Srinivasa, and R. Sukthankar, "Decentralized estimation and control of graph connectivity in mobile sensor networks," in *Proc. 2008 American Control Conference*, Seattle, WA, June 2008, pp. 2678–2683.

[45] F. Knorn, R. Stanojevic, M. Corless, and R. Shorten, "A framework for decentralized feedback connectivity control with application to sensor networks," *International Journal of Control*, vol. 82, no. 11, pp. 2095–2114, 2009.

[46] M. M. Zavlanos and G. J. Pappas, "Distributed connectivity control of mobile networks," *IEEE Transactions on Robotics*, vol. 24, no. 6, pp. 1416–1428, December 2008.

[47] ——, "Controlling connectivity of dynamic graphs," in *Proc. 44th IEEE Conference on Decision and Control and European Control Conference*, Seville, Spain, December 2005, pp. 6388–6393.

[48] K. Srivastava and M. W. Spong, "Multi-agent coordination under connectivity constraints," in *Proc. 2008 American Control Conference*, Seattle, WA, June 2008, pp. 2648–2653.

[49] G. Notarstefano, K. Savla, F. Bullo, and A. Jadbabaie, "Maintaining limited-range connectivity among second-order agents," in *Proc. 2006 American Control Conference*, Minneapolis, MN, June 2006, pp. 2124–2129.

[50] F. Bullo, J. Cortes, and S. Martinez, *Distributed Control of Robotic Networks*, ser. Applied Mathematics Series. Princeton University Press, 2009.

[51] D. P. Spanos and R. M. Murray, "Robust connectivity of networked vehicles," in *Proc. 43rd IEEE Conference on Decision and Control*, Bahamas, December 2004, pp. 2893–2898.

[52] D. V. Dimarogonas and K. J. Kyriakopoulos, "Connectedness preserving distributed swarm aggregation for multiple kinematic robots," *IEEE Transactions on Robotics*, vol. 24, no. 5, pp. 1213–1223, October 2008.

[53] A. Cornejo and N. Lynch, "Connectivity service for mobile ad-hoc networks," in *Proc. 2nd IEEE International Conference on Self-Adaptive and Self-Organizing Systems Workshops*, October 2008, pp. 292–297.

[54] Z. Yao and K. Gupta, "Backbone-based connectivity control for mobile networks," in *Proc. IEEE International Conference on Robotics and Automation*, Kobe, Japan, May 2009, pp. 1133–1139.

[55] M. M. Zavlanos, A. Jadbabaie, and G. J. Pappas, "Flocking while preserving network connectivity," in *Proc. 46th IEEE Conference on Decision and Control*, New Orleans, LA, December 2007, pp. 2919–2924.

[56] M. Ji and M. Egerstedt, "Coordination control of multi-agent systems while preserving connectedness," *IEEE Transactions on Robotics*, vol. 23, no. 4, pp. 693–703, August 2007.

[57] J. Wagenpfeil, A. Trachte, T. Hatanaka, M. Fujita, and O. Sawodny, "A distributed minimum restrictive connectivity maintenance algorithm," in *Proc. 9th International Symposium on Robot Control*, Gifu, Japan, September 2009, pp. 499–504.

[58] M. Schuresko and J. Cortes, "Distributed motion constraints for algebraic connectivity of robotic networks," *Journal of Intelligent and Robotic Systems*, vol. 56, no. 1-2, pp. 99–126, September 2009.

[59] ——, "Distributed tree rearrangements for reachability and robust connectivity," in *Hybrid Systems: Computetation and Control*, ser. Lecture Notes in Computer Science. Springer, 2009, vol. 5469, pp. 470–474.

[60] D. P. Spanos and R. M. Murray, "Motion planning with wireless network constraints," in *Proc. 2005 American Control Conference*, Portland, OR, June 2005, pp. 87–92.

[61] E. Stump, A. Jadbabaie, and V. Kumar, "Connectivity management in mobile robot teams," in *Proc. IEEE International Conference on Robotics and Automation*, Pasadena, CA, May 2008, pp. 1525–1530.

[62] A. Ganguli, J. Cortes, and F. Bullo, "Multirobot rendezvous with visibility sensors in nonconvex environments," *IEEE Transactions on Robotics*, vol. 25, no. 2, pp. 340–352, 2009.

[63] J. Cortes, S. Martinez, and F. Bullo, "Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions," *IEEE Transactions on Automatic Control*, vol. 51, no. 8, pp. 1289–1298, 2006.

[64] M. M. Zavlanos, H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Hybrid control for connectivity preserving flocking," *IEEE Transactions on Automatic Control*, vol. 54, no. 12, pp. 2869–2875, December 2009.

[65] D. V. Dimarogonas and K. H. Johansson, "Decentralized connectivity maintenance in mobile networks with bounded inputs," in *Proc. IEEE International Conference on Robotics and Automation*, Pasadena, CA, May 2008, pp. 1507–1512.

[66] A. Ajorlou and A. G. Aghdam, "A class of bounded distributed controllers for connectivity preservation of unicycles," in *Proc. 49th IEEE Conference on Decision and Control*, Atlanta, GA, December 2010, pp. 3072–3077.

[67] A. Ajorlou, A. Momeni, and A. G. Aghdam, "A class of bounded distributed control strategies for connectivity preservation in multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 12, pp. 2828–2833, December 2010.

[68] M. A. Hsieh, A. Cowley, V. Kumar, and C. Taylor, "Maintaining network connectivity and performance in robot teams," *Journal of Field Robotics*, vol. 25, no. 1-2, pp. 111–131, 2008.

[69] A. R. Wagner and R. C. Arkin, "Communication-sensitive multi-robot reconnaissance," in *Proc. IEEE International Conference on Robotics and Automation*, New Orleans, LA, 2004, pp. 2480–2487.

[70] M. Powers and T. Balch, "Value-based communication preservation for mobile robots," in *Distributed Autonomous Robotic Systems 6*, R. Alami, R. Chatila, and H. Asama, Eds. Springer Japan, 2007, pp. 327–336.

[71] Y. Mostofi, "Decentralized communication-aware motion planning in mobile networks: An information-gain approach," *Journal of Intelligent and Robotic Systems*, vol. 56, no. 1–2, pp. 233–256, 2009.

[72] P. Flocchini, G. Prencipe, N. Santoro, and P. Widmayer, "Gathering of asynchronous oblivious robots with limited visibility," *Theoretical Computer Science*, vol. 337, no. 1-3, pp. 147–168, 2005.

[73] R. C. Arkin and J. Diaz, "Line-of-sight constrained exploration for reactive multiagent robotic teams," in *Proc. 7th International Workshop on Advanced Motion Control*, Maribor, Slovenia, 2002, pp. 455–461.

[74] S. O. Anderson, R. Simmons, and D. Goldberg, "Maintaining line-of-sight communications networks between planetray rovers," in *Proc. 2003 IEEE/RSJ International Conference on Intelligent Robots and Systems*, Las Vegas, NV, 2003, pp. 2266–2272.

[75] G. Hollinger and S. Singh, "Multi-robot coordination with periodic connectivity," in *Proc. IEEE International Conference on Robotics and Automation*, 2010, (accepted).

[76] T. Gustavi, D. V. Dimarogonas, M. Egerstedt, and X. Hu, "Sufficient conditions for connectivity maintenance and rendezvous in leader-follower networks," *Automatica*, vol. 46, no. 1, pp. 133–139, January 2010.

[77] D. Tardioli, A. R. Mosteo, L. Riazuelo, J. L. Villarroel, and L. Montano, "Enforcing network connectivity in robot team missions," *International Journal of Robotics Research*, (to appear).

[78] N. Michael, M. M. Zavlanos, V. Kumar, and G. J. Pappas, "Maintaining connectivity in mobile robot networks," in *Experimental Robotics*, ser. Tracts in Advanced Robotics. Springer, 2009, pp. 117–126.

[79] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

[80] D. Kempe and F. McSherry, "A decentralized algorithm for spectral analysis," *Journal of Computer and System Sciences*, vol. 74, no. 1, pp. 70 – 83, 2008.

[81] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods for Multiagent Networks*. Princeton, NJ: Princeton University Press, 2010.

[82] A. Barbosa, "The effects of predation risk on scanning and flocking behavior in dunlin," *Journal of Field Ornithology*, vol. 68, no. 4, pp. 607–612, 1997.

[83] N. Leonard and E. Fiorelli, "Virtual leaders, artificial potentials and coordinated control of groups," in *Proc. 40th IEEE Conference on Decision and Control*, Orlando, FL, December 2001, pp. 2968–2973.

[84] R. O. Saber and R. M. Murray, "Flocking with obstacle avoidance: Cooperation with limited communication in mobile networks," in *Proc. 42nd IEEE Conference on Decision and Control*, Maui, Hawaii, December 2003, pp. 2022–2028.

[85] J. Toner and Y. Tu, "Flocks, herds and schools: A quantitative theory of flocking," *Physical Review E*, vol. 58, no. 4, pp. 4828–4858, October 1998.

[86] C. Reynolds, "Flocks, birds and schools: A distributed behavioral model," *ACM SIGGRAPH Computer Graphics*, vol. 21, no. 4, pp. 25–34, 1987.

[87] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Flocking in fixed and switching networks," *IEEE Transactions on Automatic Control*, vol. 52, no. 5, pp. 863–868, May 2007.