Distributed Primal-Dual Methods for Online Constrained Optimization

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***Abstract*— This paper introduces a decentralized primal-dual method for online distributed optimization involving global constraints. We employ a consensus-based framework and exploit the decomposability of the constraints in dual domain. At each stage, each agent commits to an adaptive decision pertaining only to the past and locally available information, and incurs a new cost function reflecting the change in the environment. We show that the algorithm achieves a regret of order $O(\sqrt{T})$ at any node with the time horizon T, in scenarios when the underlying communication topology is time-varying and jointly-connected.**

I. Introduction

Many engineering applications concerning the coordination of multi-agent systems can be cast as distributed optimization problem over networks, see e.g. [1], [2]. In these applications, distributed agents, which only have access to local parameters, often try to minimize a global cost function subject to global constraints. The main feature of any protocol designed to carry this optimization over a network is that the agents only use information received from their immediate (one-hop) neighbors. The underlying communication structure of the agents can be cast as a graph, often directed and time-varying.

A common feature in many practical applications is that they live in dynamically changing and uncertain environments. One of the existing methods which can be used to address the uncertainties arising in these problems is online optimization, where the cost function changes over time and an adaptive decision pertaining only to the past information has to be made at each stage. The objective in online optimization is to reduce *regret*, a quantity capturing the difference between the accumulated cost incurred up to some arbitrary time and the cost obtained from the best fixed point chosen in hindsight. Online optimization has been studied extensively both in the optimization and machine learning community, see e.g. [3]. Most of the online optimization algorithms are built on gradient descent methods which exploit the convexity of a cost function.

In this paper, we focus only on online distributed optimization in a large network using a *consensus* framework, where at each stage, each agent faces a new objective function reflecting the change in its nearby environment and can share its information only with neighbors. There are two distinctive ways that consensus can be used to obtain distributed optimization methods: *(A)* a decentralized weighted averaging is interleaved between optimization steps as a means of information mixing and network-wide agreement, and *(B)* the optimization problem is recast in an equivalent decomposable form by adding consistency constraints along all edges, i.e., $\mathbf{x}_i = \mathbf{x}_j$ if agent i and j are connected through an edge.

The consensus framework *(A)* was used in [4], where variants of the dual-averaging method by Nesterov [5] and the subgradient methods [6] for online distributed optimization were proposed. Other recent work includes [7], [8] based on the push-sum protocol [9], which allows for time-varying weight-imbalanced digraphs. On the other hand, the consensus framework *(B)* was used in [10], [11], where distributed online subgradient descent methods were proposed for time-varying weight-balanced digraphs [10], and a saddle-point methodwas proposed for a fixed undirected graph without any global constraints [11].

In this paper, we propose an online distributed variant of the saddle point algorithm by Arrow-Hurwicz-Uzawa [12], also known as Primal-Dual method, which is based on the consensus framework *(A)*. The main purpose of the development of this algorithm is to properly handle (time-invariant) global constraints of the form $\sum_{i=1}^{N} \mathbf{g}_i(\mathbf{x}) \preceq \mathbf{0}$, where $\mathbf{g}_i$ is only known to agent i, by exploiting the decomposability of the constraints in dual domain. We show that our developed algorithm obtains a sublinear regret over any sequence of jointly connected time-varying digraphs.

Notations: For a matrix $A \in \mathbb{R}^{n\times m}$, we use $[A]_{ij}$ to denote the entry of i-th row and j-th column. For any $N \geq 1$, the set of integers $\{1, \ldots, N\}$ is denoted by $[N]$. The symbols $\preceq$ and $\succeq$ are understood as component-wise inequalities.

II. Problem Formulation

Consider a networked system consisting of N agents indexed by the set $\mathcal{V} = [N]$. The communication among the network agents is governed by a sequence of time-varying weighted digraphs $\mathcal{G}_t = (\mathcal{V}, \mathcal{E}_t, W_t)$, for $t \geq 1$. There exists a directed link $(i, j) \in \mathcal{E}_t \subseteq \mathcal{V} \times \mathcal{V}$ if and only if agent i can receive a message from agent j at time t. Here $W_t \in \mathbb{R}^{N\times N}$ is a weight matrix respecting the topology of the graph $\mathcal{G}_t$, i.e., $[W_t]_{ij} > 0$ if and only if $(i, j) \in \mathcal{E}_t$ and $[W_t]_{ij} = 0$, otherwise. Each entry $[W_t]_{ij}$ represents a weight that agent i allocates to the incoming link (i, j). We denote the set of incoming neighbors of agent i at time t by $\mathcal{N}_{i,t} = \{j \mid (i, j) \in \mathcal{E}_t\} \cup \{i\}$, where we always include agent i itself to its own set of neighbors.

At each time t, each agent $i \in \mathcal{V}$ selects a state $\mathbf{x}_{i,t}$ from X, which is a closed and bounded subset of $\mathbb{R}^n$, based on some strategy. After this, each agent i incurs a time-varying cost $f_{i,t}$ chosen by nature which comes from a fixed class

This work is supported by NSF under grant ECCS #1543872, and by ONR under grant #N000141410479. S. Lee and M. Zavlanos are with the Department of Mechanical Engineering and Materials Science, Duke University, Durham, NC 27708, USA. emails: {s.lee,michael.zavlanos}@duke.edu

$\mathcal{F}$ of convex functions $f : X \to \mathbb{R}$. Note that the network-wide cost that agents want to minimize at time t is $f_t(\mathbf{x}) = \sum_{i=1}^N f_{i,t}(\mathbf{x})$, but f_t is not collectively known to any of the agents. In addition to this, there exist some global (time-invariant) constraint functions $\sum_{i=1}^N \mathbf{g}_i(x) \preceq \mathbf{0}$, where each $\mathbf{g}_i : \mathbb{R}^n \to \mathbb{R}^m$ is only available to agent i.

Since the cost functions $\{f_{i,t}\}_{t\geq 1}$ are only revealed to agent i and each constraint function $\mathbf{g}_i(\mathbf{x})$ is only known to agent i, each agent must determine its state $\mathbf{x}_{i,t}$ based on what it "thinks" the whole network state should be. We will postpone the details about the strategy based on which each agent i chooses such $\mathbf{x}_{i,t}$ to the next section, except to note that $\mathbf{x}_{i,t}$ depends only on the local cost functions revealed up to that time, i.e., $\{f_{i,s}\}_{s=1}^{t-1}$, and the coordination of the information received from the neighbors.

Consider the *network regret* at arbitrary $T \geq 1$:

$$\mathcal{R}(T) = \sum_{t=1}^T \sum_{i=1}^N f_{i,t}(\mathbf{x}_{i,t}) - \sum_{t=1}^T \sum_{i=1}^N f_{i,t}(\mathbf{x}^*), \quad (1)$$

which computes the difference between the summation of the local cost incurred at each agent's own state $\mathbf{x}_{i,t}$ and the best fixed point chosen from an offline and centralized view. Here $\mathbf{x}^*$ is defined as $\mathbf{x}^* \triangleq \inf_{\mathbf{x}\in X} \sum_{t=1}^T \sum_{i=1}^N f_{i,t}(\mathbf{x})$, i.e., $\mathbf{x}^*$ is a single state that could have been achieved with perfect advance knowledge of all cost functions $\{f_{i,t}\}_{i\in\mathcal{V},t\geq 1}$ and without any restriction on the communication between any agents.

A quantity of more interest is the *individual regret* as seen by agent $j \in \mathcal{V}$ at arbitrary $T \geq 1$:

$$\mathcal{R}^j(T) = \sum_{t=1}^T \sum_{i=1}^N f_{i,t}(\mathbf{x}_{j,t}) - \sum_{t=1}^T \sum_{i=1}^N f_{i,t}(\mathbf{x}^*). \quad (2)$$

This *individual regret* computes the difference between the network cost incurred by a single agent's sequence of states $\{\mathbf{x}_{j,t}\}_{t=1}^T$ and the best fixed point $\mathbf{x}^*$. Designing a coordination algorithm which reduces $\mathcal{R}^j$ for all $j \in \mathcal{V}$ is more meaningful as this would guarantee the average cost computed at any location of the network is nearly as low as the cost of the best choice $\mathbf{x}^*$.

Thus, the goal of this paper is to design a distributed update rule for each agent $i \in \mathcal{V}$ such that the individual regret (2) (a) is sublinear over the time horizon T (such that the average $\mathcal{R}^j(T)/T$ converges to zero), and (b) shows reasonable dependence on the total number of network agents and on the topology of the communication graphs.

III. Consensus-Based Online Primal-Dual

We now introduce an algorithm for online distributed constrained optimization which combines the saddle point algorithm of Arrow-Hurwicz-Uzawa [12] with a consensus framework for coordination.

In this algorithm, each agent $i \in \mathcal{V}$ generates a sequence $\{\mathbf{x}_{i,t}, \boldsymbol{\lambda}_{i,t}\}_{t\geq 1} \subseteq X \times \Lambda$, where $\Lambda \subseteq \mathbb{R}_+^m$. We assume that each agent i initializes its local primal and dual point with an arbitrary $\mathbf{x}_{i,1} \in X$ and $\boldsymbol{\lambda}_{i,1} \in \Lambda$. Then, the primal and dual iterates $\mathbf{x}_{i,t}$ and $\boldsymbol{\lambda}_{i,t}$ are updated recursively as follows:

$$\tilde{\mathbf{x}}_{i,t} = \sum_{j=1}^N [W_t]_{ij} \mathbf{x}_{j,t} \quad (3a)$$

$$\tilde{\boldsymbol{\lambda}}_{i,t} = \sum_{j=1}^N [W_t]_{ij} \boldsymbol{\lambda}_{j,t} \quad (3b)$$

$$\mathbf{x}_{i,t+1} = P_X[\tilde{\mathbf{x}}_{i,t} - \alpha_t \nabla_{\mathbf{x}} \mathcal{L}_{i,t}(\tilde{\mathbf{x}}_{i,t}, \tilde{\boldsymbol{\lambda}}_{i,t})] \quad (3c)$$

$$\boldsymbol{\lambda}_{i,t+1} = P_\Lambda[\tilde{\boldsymbol{\lambda}}_{i,t} + \alpha_t \nabla_{\boldsymbol{\lambda}} \mathcal{L}_{i,t}(\tilde{\mathbf{x}}_{i,t}, \tilde{\boldsymbol{\lambda}}_{i,t})], \quad (3d)$$

where $\{\alpha_t\}_{t\geq 1}$ is a nonincreasing sequence of positive step sizes, the Lagrangian function at time t is defined as

$$\begin{aligned}\mathcal{L}_t(\mathbf{x}, \boldsymbol{\lambda}) &= \sum_{i=1}^N f_{i,t}(\mathbf{x}) + \langle \boldsymbol{\lambda}, \sum_{i=1}^N \mathbf{g}_i(\mathbf{x}) \rangle \\ &= \sum_{i=1}^N [f_{i,t}(\mathbf{x}) + \langle \boldsymbol{\lambda}, \mathbf{g}_i(\mathbf{x}) \rangle] \triangleq \sum_{i=1}^N \mathcal{L}_{i,t}(\mathbf{x}, \boldsymbol{\lambda}),\end{aligned}$$

and its gradients at $(\tilde{\mathbf{x}}_{i,t}, \tilde{\boldsymbol{\lambda}}_{i,t})$ with respect to $\mathbf{x}$ and $\boldsymbol{\lambda}$ are represented as

$$\nabla_{\mathbf{x}} \mathcal{L}_{i,t}(\tilde{\mathbf{x}}_{i,t}, \tilde{\boldsymbol{\lambda}}_{i,t}) = \nabla f_{i,t}(\tilde{\mathbf{x}}_{i,t}) + \langle \nabla \mathbf{g}_i(\tilde{\mathbf{x}}_{i,t})', \boldsymbol{\lambda} \rangle \quad (4a)$$

$$\nabla_{\boldsymbol{\lambda}} \mathcal{L}_{i,t}(\tilde{\mathbf{x}}_{i,t}, \tilde{\boldsymbol{\lambda}}_{i,t}) = \mathbf{g}_i(\tilde{\mathbf{x}}_{i,t}). \quad (4b)$$

Here each $\nabla f_{i,t}(\tilde{\mathbf{x}}_{i,t})$ is a vector of dimension n and $\nabla \mathbf{g}_i(\tilde{\mathbf{x}}_{i,t})$ is an $m \times n$ Jacobian matrix.

The averaged vectors $\tilde{\mathbf{x}}_{i,t}$ and $\tilde{\boldsymbol{\lambda}}_{i,t}$ computed by agent i at time t will be an estimate of the network-wide primal and dual point as seen by agent i for the gradient evaluation. In consequence, the steps in (3c)-(3d) become an approximation of the centralized online primal-dual algorithm update [12]. This will become more apparent later when we bound the network wide disagreement $\|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|$ and $\|\boldsymbol{\lambda}_{i,t} - \bar{\boldsymbol{\lambda}}_t\|$ in Lemma 3, where $\bar{\mathbf{x}}_t$ and $\bar{\boldsymbol{\lambda}}_t$ denote the running average of $\mathbf{x}_{i,t}$ and $\boldsymbol{\lambda}_{i,t}$ for $i \in \mathcal{V}$, respectively, i.e., $\bar{\mathbf{x}}_t = \frac{1}{N}\sum_{i=1}^N \mathbf{x}_{i,t}$ and $\bar{\boldsymbol{\lambda}}_t = \frac{1}{N}\sum_{i=1}^N \boldsymbol{\lambda}_{i,t}$.

IV. Regret Analysis

We now provide a regret analysis of the proposed consensus-based online primal-dual method. In Sections IV-A-IV-C, we present additional assumptions and two key lemmas. The main results are provided in Section IV-D.

A. Assumptions

Our regret analysis is based on the following assumptions.

Assumption 1: (a) The functions $\{f_{i,t}\}_{i\in\mathcal{V},t\geq 1}$ and $\{\mathbf{g}_i\}_{i\in\mathcal{V}}$ are convex and continuously differentiable.

(b) The set X is compact. That is, there exists a constant $C_{\mathbf{x}}$ such that $\|\mathbf{x}\| \leq C_{\mathbf{x}}$ for all $\mathbf{x} \in X$.

(c) The set Λ onto which the dual variables are projected is included in a 2-norm ball of radius $C_{\boldsymbol{\lambda}}$, i.e.,

$$\|\boldsymbol{\lambda}\| \leq C_{\boldsymbol{\lambda}} \text{ for all } \boldsymbol{\lambda} \in \Lambda.$$

Direct consequences of Assumption 1(a) and 1(b) are gradient boundedness and Lipschitz continuity of the functions $\{f_{i,t}\}_{i\in\mathcal{V},t\geq 1}$, i.e., there exists a constant $L_f > 0$ such that

$$\|\nabla f_{i,t}(\mathbf{x})\| \leq L_f, \text{ for all } \mathbf{x} \in X, \quad (5)$$

$$|f_{i,t}(\mathbf{x}) - f_{i,t}(\mathbf{y})| \le L_f \|\mathbf{x}-\mathbf{y}\|, \text{ for all } \mathbf{x},\mathbf{y} \in X. \quad (6)$$

Similarly, for the constraint functions $\{\mathbf{g}_i\}_{i\in\mathcal{V}}$, there exists a constant $L_g > 0$ such that

$$\|\nabla \mathbf{g}_i(\mathbf{x})\| \le L_g, \text{ for all } \mathbf{x} \in X, \quad (7)$$

$$\|\mathbf{g}_i(\mathbf{x}) - \mathbf{g}_i(\mathbf{y})\| \le L_g \|\mathbf{x}-\mathbf{y}\|, \text{ for all } \mathbf{x},\mathbf{y} \in X. \quad (8)$$

Furthermore, the compactness of X and the continuity of the functions $\{f_{i,t}\}_{i\in\mathcal{V},t\ge1}$ and $\{\mathbf{g}_i\}_{i\in\mathcal{V}}$ imply that there exist constants $C_f, C_g > 0$ such that

$$|f_{i,t}(\mathbf{x})| \le C_f, \text{ for all } \mathbf{x} \in X, \quad (9)$$

$$\|\mathbf{g}_i(\mathbf{x})\| \le C_g, \text{ for all } \mathbf{x} \in X. \quad (10)$$

Assumption 2: (a) The functions $\{f_{i,t}(\mathbf{x})\}_{i\in\mathcal{V},t\ge1}$ have Lipschitz continuous gradients, i.e., there exists a constant $G_f > 0$ such that

$$\|\nabla f_{i,t}(\mathbf{x}) - \nabla f_{i,t}(\mathbf{y})\| \le G_f \|\mathbf{x}-\mathbf{y}\|, \text{ for all } \mathbf{x},\mathbf{y} \in X.$$

(b) The functions $\{\mathbf{g}_i(\mathbf{x})\}_{i\in\mathcal{V}}$ have Lipschitz continuous gradients, i.e., there exists a constant $G_g > 0$ such that

$$\|\nabla \mathbf{g}_i(\mathbf{x}) - \nabla \mathbf{g}_i(\mathbf{y})\| \le G_g \|\mathbf{x}-\mathbf{y}\|, \text{ for all } \mathbf{x},\mathbf{y} \in X.$$

Assumptions 1 and 2 are typical in any gradient-based algorithms. They ensure the boundedness of the gradients of the Lagrangian functions in (4a)-(4b) so that the gradient based method in (3a)-(3d) is well behaved.

We also make the following assumptions on the communication model.

Assumption 3: For all $t \ge 1$, the weighted graphs $\mathcal{G}_t = (\mathcal{V}, \mathcal{E}_t, W_t)$ satisfy:

(a) There exists a scalar $\eta \in (0,1)$ such that $[W_t]_{ij} \ge \eta$ if $j \in \mathcal{N}_{i,t}$. Otherwise, $[W_t]_{ij} = 0$.
(b) The weight matrix W_t is doubly stochastic, i.e., $\sum_{i=1}^N [W_t]_{ij} = 1$ for all $j \in \mathcal{V}$ and $\sum_{j=1}^N [W_t]_{ij} = 1$ for all $i \in \mathcal{V}$.
(c) There exists a scalar $Q > 0$ such that the graph $\left(\mathcal{V}, \cup_{\ell=0}^{Q-1} \mathcal{E}_{t+\ell}\right)$ is strongly connected for any $t \ge 1$.

B. Basic Iterate Relations

In this section, we provide some basic relations that hold for the sequences $\{\mathbf{x}_{i,t}\}$ and $\{\boldsymbol{\lambda}_{i,t}\}$ obtained by the distributed online primal-dual algorithm in (3a)-(3d). The relations will play an important role in our analysis of the sublinearly bounded network and individual regret. We omit its proof due to space limitations.

Lemma 1: Let Assumptions 1 and 2 hold. Consider sequences $\{\mathbf{x}_{i,t}\}_{i\in\mathcal{V},t\ge1}$ and $\{\boldsymbol{\lambda}_{i,t}\}_{i\in\mathcal{V},t\ge1}$ generated by the distributed online primal-dual algorithm in (3a)-(3d). Then, we have:

(a) For all $\mathbf{x} \in X$ and $t \ge 1$,

$$\sum_{i=1}^N [\mathcal{L}_{i,t}(\mathbf{x}_{i,t}, \boldsymbol{\lambda}_{i,t}) - \mathcal{L}_{i,t}(\mathbf{x}, \boldsymbol{\lambda}_{i,t})] \quad (11)$$
$$\le \frac{1}{2\alpha_t}\left(\sum_{i=1}^N \|\mathbf{x}_{i,t}-\mathbf{x}\|^2 - \sum_{i=1}^N \|\mathbf{x}_{i,t+1}-\mathbf{x}\|^2\right)$$
$$+ A\sum_{i=1}^N \|\tilde{\mathbf{x}}_{i,t} - \mathbf{x}_{i,t}\|$$
$$+ 2C_{\mathbf{x}}L_f \sum_{i=1}^N \|\tilde{\boldsymbol{\lambda}}_{i,t} - \boldsymbol{\lambda}_{i,t}\| + \frac{\alpha_t}{2}N(L_f + C_{\boldsymbol{\lambda}}L_g)^2,$$

where $A = 2C_{\mathbf{x}}G_f + 2C_{\mathbf{x}}C_{\boldsymbol{\lambda}}G_g + L_f + C_{\boldsymbol{\lambda}}L_g$.

(b) For all $\boldsymbol{\lambda} \in \Lambda$ and $t \ge 1$,

$$\sum_{i=1}^N [\mathcal{L}_{i,t}(\mathbf{x}_{i,t}, \boldsymbol{\lambda}) - \mathcal{L}_{i,t}(\mathbf{x}_{i,t}, \boldsymbol{\lambda}_{i,t})] \quad (12)$$
$$\le \frac{1}{2\alpha_t}\left(\sum_{i=1}^N \|\boldsymbol{\lambda}_{i,t}-\boldsymbol{\lambda}\|^2 - \sum_{i=1}^N \|\boldsymbol{\lambda}_{i,t+1}-\boldsymbol{\lambda}\|^2\right)$$
$$+ \frac{\alpha_t}{2}NC_g^2 + C_g\sum_{i=1}^N \|\tilde{\boldsymbol{\lambda}}_{i,t} - \boldsymbol{\lambda}_{i,t}\|$$
$$+ 2C_{\boldsymbol{\lambda}}L_g \sum_{i=1}^N \|\tilde{\mathbf{x}}_{i,t} - \mathbf{x}_{i,t}\|.$$

C. Sublinear Network Disagreement

We will employ a result from [13]. It shows that a set of sequences recursively updated by weighted averaging followed by perturbation will not drift too much from its mean field as long as the underlying weight matrices satisfy the properties listed in Assumption 3 and the perturbation behaves nicely. It also exhibits that the convergence speed depends on the minimum weight η in the averaging matrix, the number of nodes N, and the topology of the communication graphs Q.

Lemma 2: Let Assumption 3 hold for a sequence of weight matrices $\{W_t\}_{t\ge1}$. Consider a set of sequences $\{\boldsymbol{\theta}_{i,t}\}$ for $i \in [N]$ defined by the following relation:

$$\boldsymbol{\theta}_{i,t+1} = \sum_{j=1}^N [W_t]_{ij}\boldsymbol{\theta}_{j,t} + \boldsymbol{\epsilon}_{i,t+1}, \text{ for } t \ge 1. \quad (13)$$

Let $\bar{\boldsymbol{\theta}}_t$ denote the running average of $\boldsymbol{\theta}_{i,t}$ for $i \in [N]$, i.e., $\bar{\boldsymbol{\theta}}_t = \frac{1}{N}\sum_{i=1}^N \boldsymbol{\theta}_{i,t}$. Then,

$$\|\boldsymbol{\theta}_{i,t+1} - \bar{\boldsymbol{\theta}}_{t+1}\| \le N\gamma\beta^t \max_j \|\boldsymbol{\theta}_{j,1}\| \quad (14)$$
$$+ \gamma\sum_{\ell=1}^{t-1}\beta^{t-\ell}\sum_{j=1}^N \|\boldsymbol{\epsilon}_{j,\ell+1}\| + \frac{1}{N}\sum_{j=1}^N \|\boldsymbol{\epsilon}_{j,t+1}\| + \|\boldsymbol{\epsilon}_{i,t+1}\|,$$

where γ and β are defined as

$$\gamma = \left(1 - \frac{\eta}{2N^2}\right)^{-2} \quad \beta = \left(1 - \frac{\eta}{2N^2}\right)^{\frac{1}{Q}}. \quad (15)$$

The following corollary shows that if the perturbation can be bounded and controlled by a decreasing step size, the discrepancy of each sequence from its mean field summed over an arbitrary time horizon can be sublinearly bounded.

Corollary 1: Suppose that for the sequences $\{\boldsymbol{\theta}_{i,t}\}$ defined in Lemma 2 there exist a positive scalar sequence $\{\alpha_t\}_{t\ge1}$ and a constant $K > 0$ such that $\|\boldsymbol{\epsilon}_{i,t}\| \le K\alpha_t$ for all $i \in [N]$ and $t \ge 1$. Then, for any $T \ge 2$:

$$\sum_{t=1}^{T-1} \|\boldsymbol{\theta}_{i,t+1} - \bar{\boldsymbol{\theta}}_{t+1}\| \le \frac{\gamma N\beta}{1-\beta}\max_j \|\boldsymbol{\theta}_{j,1}\|$$

$$+ \left(\frac{\gamma N K\beta}{1-\beta} + 2K\right) \sum_{t=1}^{T-1} \alpha_t,$$

where γ and β are defined in (15).

Proof: By summing the relation (14) over $t = 1, \ldots, T-1$, we obtain

$$\sum_{t=1}^{T-1} \|\boldsymbol{\theta}_{i,t+1} - \bar{\boldsymbol{\theta}}_{t+1}\| \le N\gamma \max_j \|\boldsymbol{\theta}_{j,1}\| \sum_{t=1}^{T-1} \beta^t \quad (16)$$
$$+ \gamma \sum_{t=1}^{T-1}\sum_{\ell=1}^{t-1} \beta^{t-\ell} \sum_{j=1}^{N} \|\boldsymbol{\epsilon}_{j,\ell+1}\|$$
$$+ \frac{1}{N}\sum_{t=1}^{T-1}\sum_{j=1}^{N} \|\boldsymbol{\epsilon}_{j,t+1}\| + \sum_{t=1}^{T-1} \|\boldsymbol{\epsilon}_{i,t+1}\|.$$

By rearranging the summations, the second term on the right-hand side of the above relation can be bounded as:

$$\sum_{t=1}^{T-1}\sum_{\ell=1}^{t-1} \beta^{t-\ell} \sum_{j=1}^{N} \|\boldsymbol{\epsilon}_{j,\ell+1}\| \le NK \sum_{t=1}^{T-1}\sum_{\ell=1}^{t-1} \beta^{t-\ell}\alpha_t$$
$$= NK \sum_{\ell=1}^{T-2} \beta^{\ell} \sum_{t=1}^{T-\ell-1} \alpha_t \le NK \sum_{\ell=1}^{T-2} \beta^{\ell} \sum_{t=1}^{T-1} \alpha_t. \quad (17)$$

Using $\sum_{t=1}^{T-1} \beta^t \le \frac{\beta}{1-\beta}$, relation (17), and the assumption $\|\boldsymbol{\epsilon}_{i,t}\| \le K\alpha_t$ to upper bound inequality (16), we obtain the desired result. ■

Note that our primal iterates $\mathbf{x}_{i,t}$ in algorithm (3c) can be represented by relation (13). Indeed, (3c) can be rewritten as

$$\mathbf{x}_{i,t+1} = \sum_{j=1}^{N} [W_t]_{ij} \mathbf{x}_{i,t} + \boldsymbol{\epsilon}_{i,t+1},$$

with

$$\boldsymbol{\epsilon}_{i,t+1} = P_X[\tilde{\mathbf{x}}_{i,t} - \alpha_t \nabla_{\mathbf{x}} \mathcal{L}_{i,t}(\tilde{\mathbf{x}}_{i,t}, \tilde{\boldsymbol{\lambda}}_{i,t})] - \tilde{\mathbf{x}}_{i,t}. \quad (18)$$

Similarly, our primal iterates $\boldsymbol{\lambda}_{i,t}$ in algorithm (3d) can be rewritten as

$$\boldsymbol{\lambda}_{i,t+1} = \sum_{j=1}^{N} [W_t]_{ij} \boldsymbol{\lambda}_{i,t} + \boldsymbol{\epsilon}_{i,t+1},$$

with

$$\boldsymbol{\epsilon}_{i,t+1} = P_\Lambda[\tilde{\boldsymbol{\lambda}}_{i,t} + \alpha_t \nabla_{\boldsymbol{\lambda}} \mathcal{L}_{i,t}(\tilde{\mathbf{x}}_{i,t}, \tilde{\boldsymbol{\lambda}}_{i,t})] - \tilde{\boldsymbol{\lambda}}_{i,t}. \quad (19)$$

Using this fact, we make use of Corollary 1 in the following lemma to bound the disagreement between each agent's primal and dual iterate and their corresponding mean fields.

Lemma 3: Let Assumptions 1 - 3 hold. Consider sequences $\{\mathbf{x}_{i,t}\}_{i\in\mathcal{V},t\ge1}$ and $\{\boldsymbol{\lambda}_{i,t}\}_{i\in\mathcal{V},t\ge1}$ generated by the distributed online primal-dual algorithm in (3a)-(3d). Let $\bar{\mathbf{x}}_t$ and $\bar{\boldsymbol{\lambda}}_t$ denote the running average of $\mathbf{x}_{i,t}$ and $\boldsymbol{\lambda}_{i,t}$ for $i \in \mathcal{V}$, i.e., $\bar{\mathbf{x}}_t = \frac{1}{N}\sum_{i=1}^{N} \mathbf{x}_{i,t}$ and $\bar{\boldsymbol{\lambda}}_t = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{\lambda}_{i,t}$. Then, with a stepsize choice of $\alpha_t = \frac{1}{\sqrt{t}}$, the network disagreement terms can be upper bounded as follows:

(a) For all $T \ge 1$, we have

$$\sum_{t=1}^{T}\sum_{i=1}^{N} \|\tilde{\mathbf{x}}_{i,t} - \mathbf{x}_{i,t}\| \le 2\sum_{t=1}^{T}\sum_{i=1}^{N} \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|,$$

$$\sum_{t=1}^{T}\sum_{i=1}^{N} \|\tilde{\boldsymbol{\lambda}}_{i,t} - \boldsymbol{\lambda}_{i,t}\| \le 2\sum_{t=1}^{T}\sum_{i=1}^{N} \|\boldsymbol{\lambda}_{i,t} - \bar{\boldsymbol{\lambda}}_t\|.$$

(b) For all $i \in \mathcal{V}$ and $T \ge 1$, we have

$$\sum_{t=1}^{T} \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\| \le \left(2 + \frac{\gamma N\beta}{1-\beta}\right) C_{\mathbf{x}}$$
$$+ 2(L_f + C_{\boldsymbol{\lambda}} L_g)\left(\frac{\gamma N\beta}{1-\beta} + 2\right)\sqrt{T},$$

$$\sum_{t=1}^{T} \|\boldsymbol{\lambda}_{i,t} - \bar{\boldsymbol{\lambda}}_t\| \le \left(2 + \frac{\gamma N\beta}{1-\beta}\right) C_{\boldsymbol{\lambda}}$$
$$+ 2C_g\left(\frac{\gamma N\beta}{1-\beta} + 2\right)\sqrt{T}.$$

Proof:

(a) We have the following chain of relations:

$$\sum_{i=1}^{N} \|\tilde{\mathbf{x}}_{i,t} - \mathbf{x}_{i,t}\| \le \sum_{i=1}^{N}\sum_{j=1}^{N} [W_t]_{ij} \|\mathbf{x}_{j,t} - \mathbf{x}_{i,t}\| \quad (20a)$$
$$\le \sum_{i=1}^{N}\sum_{j=1}^{N} [W_t]_{ij} [\|\mathbf{x}_{j,t} - \bar{\mathbf{x}}_t\| + \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|] \quad (20b)$$
$$= \sum_{j=1}^{N} \|\mathbf{x}_{j,t} - \bar{\mathbf{x}}_t\| \sum_{i=1}^{N} [W_t]_{ij} + \sum_{i=1}^{N} \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\| \sum_{j=1}^{N} [W_t]_{ij} \quad (20c)$$
$$= 2\sum_{i=1}^{N} \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|, \quad (20d)$$

where (20a) follows from the definition of $\tilde{\mathbf{x}}_{i,t}$ in (3a) and the convexity of the norm function and the doubly stochasticity of W_t; (20b) follows from the triangle inequality; (20c) follows from reordering of the summations; and (20d) follows from the doubly stochasticity of W_t. The result on the dual iterates follows from the exactly same line of arguments.

(b) In what follows, we estimate $\|\boldsymbol{\epsilon}_{i,t+1}\|$ in (18) to make use of Corollary 1:

$$\|\boldsymbol{\epsilon}_{i,t+1}\| = \left\| P_X[\tilde{\mathbf{x}}_{i,t} - \alpha_t \nabla_{\mathbf{x}} \mathcal{L}_{i,t}(\tilde{\mathbf{x}}_{i,t}, \tilde{\boldsymbol{\lambda}}_{i,t})] - \tilde{\mathbf{x}}_{i,t} \right\|$$
$$\le \alpha_t \left\| \nabla_{\mathbf{x}} \mathcal{L}_{i,t}(\tilde{\mathbf{x}}_{i,t}, \tilde{\boldsymbol{\lambda}}_{i,t})] \right\|$$
$$= \alpha_t \left\| \nabla f(\tilde{\mathbf{x}}_{i,t}) + \langle \nabla \mathbf{g}_i(\tilde{\mathbf{x}}_{i,t})', \tilde{\boldsymbol{\lambda}}_{i,t} \rangle \right\|$$
$$\le \alpha_t (L_f + C_{\boldsymbol{\lambda}} L_g),$$

where the inequalities follow from the fact that $\tilde{\mathbf{x}}_{i,t} \in X$, the nonexpansiveness of the projection operator $P_X[\cdot]$ and the bounded gradient assumption (Assumption 1), respectively. Now we can invoke Corollary 1 with $\boldsymbol{\theta}_{i,t} := \mathbf{x}_{i,t}$ and $K := L_f + C_{\boldsymbol{\lambda}} L_g$. Hence, we have for all $i \in \mathcal{V}$

$$\sum_{t=1}^{T-1} \|\mathbf{x}_{i,t+1} - \bar{\mathbf{x}}_{t+1}\| \le \frac{\gamma N\beta}{1-\beta} \max_j \|\mathbf{x}_{j,1}\|$$
$$+ (L_f + C_{\boldsymbol{\lambda}} L_g)\left(\frac{\gamma N\beta}{1-\beta} + 2\right) \sum_{t=1}^{T-1} \alpha_t,$$

and consequently

$$\sum_{t=1}^{T}\|\mathbf{x}_{i,t}-\bar{\mathbf{x}}_t\| \leq \|\mathbf{x}_{i,1}-\bar{\mathbf{x}}_1\| + \frac{\gamma N\beta}{1-\beta}\max_j \|\mathbf{x}_{j,1}\| + (L_f + C_{\boldsymbol{\lambda}} L_g)\left(\frac{\gamma N\beta}{1-\beta}+2\right)\sum_{t=1}^{T}\alpha_t. \tag{21}$$

From the definition of the step size α_t, we have

$$\sum_{t=1}^{T}\alpha_t = \sum_{t=1}^{T}\frac{1}{\sqrt{t}} \leq 1 + \int_1^T \frac{1}{\sqrt{t}}dt \leq 2\sqrt{T}-1.$$

Combining this with relation (21) and using Assumption 1(b), we obtain the desired result.
Similarly, the error $\|\boldsymbol{\epsilon}_{i,t+1}\|$ in (19) can be estimated as follows:

$$\|\boldsymbol{\epsilon}_{i,t+1}\| = \left\|P_\Lambda[\tilde{\boldsymbol{\lambda}}_{i,t} + \alpha_t \nabla_{\boldsymbol{\lambda}}\mathcal{L}_{i,t}(\tilde{\mathbf{x}}_{i,t},\tilde{\boldsymbol{\lambda}}_{i,t})] - \tilde{\boldsymbol{\lambda}}_{i,t}\right\| \leq \alpha_t \left\|\nabla_{\boldsymbol{\lambda}}\mathcal{L}_{i,t}(\tilde{\mathbf{x}}_{i,t},\tilde{\boldsymbol{\lambda}}_{i,t})]\right\| = \alpha_t\|\mathbf{g}_i(\tilde{\mathbf{x}}_{i,t})\| \leq \alpha_t C_g,$$

where the inequalities follow from the fact that $\tilde{\boldsymbol{\lambda}}_{i,t} \in \Lambda$, the nonexpansiveness of the projection operator $P_X[\cdot]$ and (10), respectively. By invoking Corollary 1 with $\boldsymbol{\theta}_{i,t} := \boldsymbol{\lambda}_{i,t}$ and $K := C_g$ and using the exactly same line of arguments in estimating $\sum_{t=1}^{T}\|\mathbf{x}_{i,t}-\bar{\mathbf{x}}_t\|$, the desired result follows. ■

D. Main Results

In this section, we obtain sublinear bounds for the network and individual regret defined in (1) and (2), respectively.

Theorem 1: (Network Regret Bound) Let Assumptions 1 - 3 hold. Consider sequences $\{\mathbf{x}_{i,t}\}_{i\in\mathcal{V},t\geq 1}$ and $\{\boldsymbol{\lambda}_{i,t}\}_{i\in\mathcal{V},t\geq 1}$ generated by the distributed online primal-dual algorithm in (3a)-(3d). Then, with a stepsize choice of $\alpha_t = \frac{1}{\sqrt{t}}$, the network regret $\mathcal{R}(T)$ with respect to the best fixed offline decision $\mathbf{x}^* \in X$ can be upper bounded as follows: For any $T \geq 1$,

$$\mathcal{R}(T) \leq N\left(\frac{\gamma N\beta}{1-\beta}+2\right)K_1 + \left[K_2 + K_3\left(\frac{\gamma N\beta}{1-\beta}+2\right)\right]N\sqrt{T},$$

where

$$K_1 = 2C_{\mathbf{x}}(2C_{\mathbf{x}}G_f + 2C_{\mathbf{x}}C_{\boldsymbol{\lambda}}G_g + L_f + 3C_{\boldsymbol{\lambda}}L_g) + 2C_{\boldsymbol{\lambda}}(2C_{\mathbf{x}}L_f + C_g), \tag{22a}$$

$$K_2 = 2(C_{\mathbf{x}}^2 + C_{\boldsymbol{\lambda}}^2) + \left[(L_f + C_{\boldsymbol{\lambda}}L_g)^2 + C_g^2\right] \tag{22b}$$

$$K_3 = 4(L_f + C_{\boldsymbol{\lambda}}L_g)(2C_{\mathbf{x}}G_f + 2C_{\mathbf{x}}C_{\boldsymbol{\lambda}}G_g + L_f + 3C_{\boldsymbol{\lambda}}L_g) + 4C_g(2C_{\mathbf{x}}L_f + C_g). \tag{22c}$$

Proof: By adding inequality (11) and (12) in Lemma 1, and summing this over $t = 1,\ldots,T$, we have for any $\mathbf{x} \in X$ and $\boldsymbol{\lambda} \in \Lambda$

$$\sum_{t=1}^{T}\sum_{i=1}^{N}[\mathcal{L}_{i,t}(\mathbf{x}_{i,t},\boldsymbol{\lambda}) - \mathcal{L}_{i,t}(\mathbf{x},\boldsymbol{\lambda}_{i,t})] \tag{23}$$

$$\begin{aligned}\leq &\sum_{t=1}^{T}\frac{1}{2\alpha_t}\left(\sum_{i=1}^{N}\|\mathbf{x}_{i,t}-\mathbf{x}\|^2 - \sum_{i=1}^{N}\|\mathbf{x}_{i,t+1}-\mathbf{x}\|^2\right)\\ &+ \sum_{t=1}^{T}\frac{1}{2\alpha_t}\left(\sum_{i=1}^{N}\|\boldsymbol{\lambda}_{i,t}-\boldsymbol{\lambda}\|^2 - \sum_{i=1}^{N}\|\boldsymbol{\lambda}_{i,t+1}-\boldsymbol{\lambda}\|^2\right)\\ &+ a_1\sum_{t=1}^{T}\alpha_t + a_2\sum_{t=1}^{T}\sum_{i=1}^{N}\|\tilde{\mathbf{x}}_{i,t}-\mathbf{x}_{i,t}\|\\ &+ a_3\sum_{t=1}^{T}\sum_{i=1}^{N}\|\tilde{\boldsymbol{\lambda}}_{i,t}-\boldsymbol{\lambda}_{i,t}\|,\end{aligned}$$

where $a_1 = \frac{1}{2}N(L_f + C_{\boldsymbol{\lambda}}L_g)^2 + \frac{1}{2}NC_g^2$, $a_2 = 2C_{\mathbf{x}}G_f + 2C_{\mathbf{x}}C_{\boldsymbol{\lambda}}G_g + L_f + 3C_{\boldsymbol{\lambda}}L_g$ and $a_3 = 2C_{\mathbf{x}}L_f + C_g$. The first term on the right-hand side of (23) can be bounded as

$$\begin{aligned}&\sum_{t=1}^{T}\frac{1}{2\alpha_t}\left(\sum_{i=1}^{N}\|\mathbf{x}_{i,t}-\mathbf{x}\|^2 - \sum_{i=1}^{N}\|\mathbf{x}_{i,t+1}-\mathbf{x}\|^2\right)\\ &\leq \frac{1}{2\alpha_1}\sum_{i=1}^{N}\|\mathbf{x}_{i,1}-\mathbf{x}\|^2 + \frac{1}{2}\sum_{t=2}^{T}\left(\frac{1}{\alpha_t}-\frac{1}{\alpha_{t-1}}\right)\sum_{i=1}^{N}\|\mathbf{x}_{i,t}-\mathbf{x}\|^2 \leq \frac{2}{\alpha_T}NC_{\mathbf{x}}^2,\end{aligned}$$

where we dropped out a negative term $-\frac{1}{2\alpha_T}\sum_{i=1}^{N}\|\mathbf{x}_{i,T+1}-\mathbf{x}\|^2$ in the first inequality and the second inequality follows from the boundedness of the set X. Similarly, the second term on the right-hand side of (23) can be bounded as

$$\sum_{t=1}^{T}\frac{1}{2\alpha_t}\left(\sum_{i=1}^{N}\|\boldsymbol{\lambda}_{i,t}-\boldsymbol{\lambda}\|^2 - \sum_{i=1}^{N}\|\boldsymbol{\lambda}_{i,t+1}-\boldsymbol{\lambda}\|^2\right) \leq \frac{2}{\alpha_T}NC_{\boldsymbol{\lambda}}^2.$$

Combining these two relations in (23), we obtain

$$\begin{aligned}&\sum_{t=1}^{T}\sum_{i=1}^{N}\left[\mathcal{L}_{i,t}(\mathbf{x}_{i,t},\boldsymbol{\lambda}) - \mathcal{L}_{i,t}(\mathbf{x},\tilde{\boldsymbol{\lambda}}_{i,t})\right] \qquad (24)\\ &\leq \frac{2}{\alpha_T}N(C_{\mathbf{x}}^2 + C_{\boldsymbol{\lambda}}^2) + a_1\sum_{t=1}^{T}\alpha_t\\ &+ a_2\sum_{t=1}^{T}\sum_{i=1}^{N}\|\tilde{\mathbf{x}}_{i,t}-\mathbf{x}_{i,t}\| + a_3\sum_{t=1}^{T}\sum_{i=1}^{N}\|\tilde{\boldsymbol{\lambda}}_{i,t}-\boldsymbol{\lambda}_{i,t}\|.\end{aligned}$$

We further estimate the last term on the right-hand side using Lemma 3 to obtain

$$a_2\sum_{t=1}^{T}\sum_{i=1}^{N}\|\tilde{\mathbf{x}}_{i,t}-\mathbf{x}_{i,t}\| + a_3\sum_{t=1}^{T}\sum_{i=1}^{N}\|\tilde{\boldsymbol{\lambda}}_{i,t}-\boldsymbol{\lambda}_{i,t}\| \leq a_4 + a_5\sqrt{T},$$

where $a_4 = 2N\left(2+\frac{\gamma N\beta}{1-\beta}\right)(a_2C_{\mathbf{x}} + a_3C_{\boldsymbol{\lambda}})$ and $a_5 = 4N(a_2(L_f + C_{\boldsymbol{\lambda}}L_g) + a_3C_g)\left(\frac{\gamma N\beta}{1-\beta}+2\right)$. Combining this relation into (24) and using the definition of the step size α_t, we further obtain

$$\sum_{t=1}^{T}\sum_{i=1}^{N}\left[\mathcal{L}_{i,t}(\mathbf{x}_{i,t},\boldsymbol{\lambda}) - \mathcal{L}_{i,t}(\mathbf{x},\tilde{\boldsymbol{\lambda}}_{i,t})\right]$$

$$\leq a_4 + \left[2N(C_{\mathbf{x}}^2 + C_{\boldsymbol{\lambda}}^2) + 2a_1 + a_5\right]\sqrt{T}, \tag{25}$$

where we used $\sum_{t=1}^{T} \alpha_t = \sum_{t=1}^{T} \frac{1}{\sqrt{t}} \leq 1 + \int_1^T \frac{1}{\sqrt{t}} dt \leq 2\sqrt{T} - 1$. Since the above inequality holds for any $\mathbf{x} \in X$ and $\boldsymbol{\lambda} \in \Lambda$, we now let $\mathbf{x} \triangleq \mathbf{x}^* \in X$ and $\boldsymbol{\lambda} \triangleq \mathbf{0} \in \Lambda$. Since $\mathbf{g}(\mathbf{x}^*) \leq \mathbf{0}$ and $\tilde{\boldsymbol{\lambda}}_{i,t} \in \Lambda$ for each $i \in \mathcal{V}$, we have $\sum_{i=1}^{N} \langle \tilde{\boldsymbol{\lambda}}_{i,t}, \mathbf{g}(\mathbf{x}^*)\rangle \leq 0$. From this we obtain

$$\begin{aligned}
&\sum_{t=1}^{T}\sum_{i=1}^{N}\left[\mathcal{L}_{i,t}(\mathbf{x}_{i,t}, \mathbf{0}) - \mathcal{L}_{i,t}(\mathbf{x}^*, \tilde{\boldsymbol{\lambda}}_{i,t})\right] \\
&= \sum_{t=1}^{T}\sum_{i=1}^{N}\left[f_{i,t}(\mathbf{x}_{i,t}) + \langle \mathbf{0}, \mathbf{g}(\mathbf{x}_{i,t})\rangle - f_{i,t}(\mathbf{x}) - \langle \tilde{\boldsymbol{\lambda}}_{i,t}, \mathbf{g}(\mathbf{x}^*)\rangle\right] \\
&\geq \sum_{t=1}^{T}\sum_{i=1}^{N}\left[f_{i,t}(\mathbf{x}_{i,t}) - f_{i,t}(\mathbf{x}^*)\right] = \mathcal{R}(T),
\end{aligned}$$

which gives the desired result. ■

Using the previous theorem, we provide an upper bound on the individual regrets in the following theorem.

Theorem 2: (Individual Regret Bound) Let Assumptions 1 - 3 hold. Consider sequences $\{\mathbf{x}_{i,t}\}_{i\in\mathcal{V},t\geq 1}$ and $\{\boldsymbol{\lambda}_{i,t}\}_{i\in\mathcal{V},t\geq 1}$ generated by the distributed online primal-dual algorithm in (3a)-(3d). Then, with a stepsize choice of $\alpha_t = \frac{1}{\sqrt{t}}$, agent j's local regret $\mathcal{R}^j(T)$, with respect to the best fixed offline decision $\mathbf{x}^* \in X$ can be upper bounded as follows: For any $j \in \mathcal{V}$ and $T \geq 1$,

$$\begin{aligned}
\mathcal{R}^j(T) &\leq N\left(\frac{\gamma N\beta}{1-\beta} + 2\right)(K_1 + 2L_f C_{\mathbf{x}}) \\
&+ \left[K_2 + (K_3 + 4L_f^2 + 4C_{\boldsymbol{\lambda}} L_f L_g)\left(\frac{\gamma N\beta}{1-\beta} + 2\right)\right] N\sqrt{T},
\end{aligned}$$

where K_1, K_2 and K_3 are constants defined in (22a)-(22c).

Proof: By adding and subtracting $\sum_{t=1}^{T}\sum_{i=1}^{N} f_{i,t}(\mathbf{x}_{i,t})$ from $\mathcal{R}^j(T)$, we obtain

$$\begin{aligned}
&\sum_{t=1}^{T}\sum_{i=1}^{N}\left[f_{i,t}(\mathbf{x}_{j,t}) - f_{i,t}(\mathbf{x}^*)\right] \qquad (26)\\
&\leq \sum_{t=1}^{T}\sum_{i=1}^{N}\left[f_{i,t}(\mathbf{x}_{j,t}) - f_{i,t}(\mathbf{x}_{i,t})\right] + \sum_{t=1}^{T}\sum_{i=1}^{N}\left[f_{i,t}(\mathbf{x}_{i,t}) - f_{i,t}(\mathbf{x}^*)\right].
\end{aligned}$$

For the first term on the right-hand side, we have the following chain of relations:

$$\begin{aligned}
&\sum_{t=1}^{T}\sum_{i=1}^{N}\left[f_{i,t}(\mathbf{x}_{j,t}) - f_{i,t}(\mathbf{x}_{i,t})\right] \\
&\leq L_f \sum_{t=1}^{T}\sum_{i=1}^{N}\|\mathbf{x}_{j,t} - \mathbf{x}_{i,t}\| \qquad (27a)\\
&\leq L_f \sum_{i=1}^{N}\left[\sum_{t=1}^{T}\|\mathbf{x}_{j,t} - \bar{\mathbf{x}}_t\| + \sum_{t=1}^{T}\|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|\right] \qquad (27b)\\
&\leq 2NL_f\left(2 + \frac{\gamma N\beta}{1-\beta}\right)C_{\mathbf{x}} \qquad (27c)\\
&\quad + 4NL_f(L_f + C_{\boldsymbol{\lambda}} L_g)\left(\frac{\gamma N\beta}{1-\beta} + 2\right)\sqrt{T},
\end{aligned}$$

where (27a) follows from Assumption 1; (27b) follows from the triangle inequality; and (27c) follows from Lemma 3(b). From this and using Theorem 1 for bounding the second term on the right-hand side of (26), we obtain the desired result. ■

The above result captures the dependency of the individual regret on the number of agents N, the minimum weight η and the connectivity Q which are reflected in the constants γ and β. Note that when N and Q are bigger and η is smaller, β goes closer to one.

V. Conclusions and Future Work

We developed a new decentralized primal-dual method for online distributed optimization involving global constraints. Under the assumption that the cost and constraint functions, and their gradients are Lipschitz continuous, we showed that the algorithm achieves worst-case individual regret of order $O(\sqrt{T})$ on any sequence of time-varying, weight-balanced and jointly connected graphs. Future work will include numerical analysis to complement the theoretical results as well as the analysis of the regret in constraint violation.

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